Conference on Number Theory Pullman
Proc. Washington State University, 1971, p 13-21

SOME PROBLEMS ON THE PRIME FACTORS OF CONSECUTIVE INTEGERS II

by

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G. A. Grimm [3] stated the following interesting conjecture: Let $n+1,\ldots,n+k$ be consecutive composite numbers. Then for each i, $1 \le i \le k$ there is a p_i, $p_i \mid n+i$ $p_{i_1} \ne p_{i_2}$ for $i_1 \ne i_2$. He also expressed the conjecture in a weaker form stating that any set of k consecutive composite numbers need to have at least k prime factors. We first show that even in this weaker form the conjecture goes far beyond what is known about primes at present.

First we define a few number theoretic functions. Denote by $\nu(n, k)$ the number of distinct prime factors of $(n+1)\ldots(n+k)$. $f_1(n)$ is the smallest integer k so that for every $1 \le \ell \le k$

$$\nu(n, \ell) \ge \ell \text{ but } \nu(n, k+1) = k .$$

$f_0(n)$ is the largest integer k for which

$$\nu(n, k) \ge k .$$

Clearly $f_0(n) \ge f_1(n)$ and we shall show that infinitely often $f_0(n) > f_1(n)$.

Following Grimm let $f_2(n)$ be the largest integer k so that for each $1 \le i \le k$ there is a $p_i \mid n+i$, $p_{i_1} \ne p_{i_2}$ if $i_1 \ne i_2$.

Denote by $P(m)$ the greatest prime factor of m. $f_3(n)$ is the greatest integer so that all the primes $P(n+i)$, $1 \le i \le k$ are distinct. $f_4(n)$ is the largest integer k so that $P(n+i) \ge i$, $1 \le i \le k$ and $f_5(n)$ is the largest integer k so that $P(n+i) \ge k$ for every $1 \le i \le k$. Clearly

$$f_0(n) \geq f_1(n) \geq f_2(n) \geq f_3(n) \geq f_4(n) \geq f_5(n).$$

CONJECTURE: It seems certain to us that for infinitely many n the inequalities are all strict. For example, for $n = 9701$

$$f_0(n) = 96 > 94 > 90 > 45 > 18 > 11 = f_5(n) .$$

It seems very difficult to get exact information on these functions which probably behave very irregularly. By a well known theorem of Pólya, $f_3(n)$ tends to infinity. First we prove

THEOREM 1.

(1) $$f_0(n) < c_1 \left[\frac{n}{\log n}\right]^{1/2}$$

To prove (1) assume that $\nu(n, k) \geq k$. We then would have

(2) $$\binom{n+k}{k} \geq \prod p_r \ , \ \pi(k) < r \leq k$$

where $p_1 = 2 < p_2 < \ldots$ is the sequence of consecutive primes. On the other hand

(3) $$\binom{n+k}{k} < \frac{(n+k)^k}{k!} < \left(\frac{e(n+k)}{k}\right)^k .$$

A well known theorem of Rosser and Schoenfeld [4] states that for large t

(4) $$p_t > t \log t + t \log \log t - c_2 t$$

where c_1 , c_2 ... are positive absolute constants.

From (4) we obtain by a simple computation that ($\exp z = e^z$).

(5) $$\prod_{r=\pi(k)+1}^{k} p_r > \exp\,(k \log k + k \log \log k - c_3 k).$$

From (2), (3), (4) and (5) we have

(6) $$\frac{e(n+k)}{k} > k \log k / e^{c_3} .$$

(6) immediately implies (1) and the proof of Theorem 1 is complete.

We conjecture

$$f_0(n) < n^{1/2-c_4}$$

for all $n > n_0(c_4)$, perhaps $f_1(n) > n^{c_5}$ for all n. $f_0(n) < n^{1/2-c_4}$ seems to follow from a recent result of Ramachandra (A note on numbers with a large prime factor, Journal London Math. Soc. 1 (1969), pp. 303-306) but we do not give the details here.

Theorem 1 shows that there is not much hope to prove Grimm's conjecture in the "near future" since even its weaker form implies that

$$p_{i+1} - p_i < c(p_i / \log p_i)^{1/2}$$

in particular it would imply that there are primes between n^2 and $(n + 1)^2$ for all sufficiently large n.

Next we show

THEOREM 2. For infinitely many n

$$f_0(n) < c_6 n^{1/e} \text{ and } f_1(n) < c_7 n^{1/e} .$$

Denote by $u(m, X)$ the number of prime factors of m in $(c_8 X^{1/e}, X)$. We evidently have

(7) $$\sum_{m=1}^{X} u(m, X) = \sum_{c_8 X^{1/e} < p < X} [\frac{X}{p}] > X \sum_{c_8 X^{1/e} < p < X} \frac{1}{p} - \pi(X) > X$$

for sufficiently small c_8.

From (7) it is easy to see that there is an $c_8X^{1/e} \leq m < X - c_8X^{1/e}$ so that for every $t \leq X - m$

$$\sum_{i=1}^{t} u(m + i, X) \geq t .$$

Choose $t = c_6X^{1/e}$ and we obtain Theorem 2. In fact for every $t < c_6X^{1/e}$ $\prod_{i=1}^{t} (m + i)$ has at least t prime factors $> c_6X^{1/e}$. The same method gives that $f_1(n) < c_7n^{1/e}$ holds for infinitely many n.

We can improve a result of Grimm by

THEOREM 3.* For every $n > n_o$

$$f_2(n) > (1 + o(1)) \log n .$$

Suppose $f_2(n) < t$. This implies by Hall's theorem that for some $r \leq \pi(t)$ there are r primes $p_1,\ldots,p_r$ so that $r + 1$ integers $n + i_1,\ldots,n + i_{r+1}$, $1 \leq i_1 < \ldots < i_{r+1} \leq t$ are entirely composed of $p_1,\ldots,p_r$. For each p there is at most one of the integers $n + j$, $1 < j \leq t$ which divide p^α with $p^\alpha > t$. Thus for at least one index i_s, $1 \leq s \leq r + 1$

$$n + i_s = \prod_{i=1}^{t} p_i^{\alpha_i} , \; p_i^{\alpha_i} < t, \text{ or } n < t^{\pi(r)} < t^{\pi(t)} < e^{(1+o(1))t}$$

which proves Theorem 3. Probably this proof can be improved to give $f_2(n) / \log n \to \infty$ but at the moment we can not see how to get $f_2(n) > (\log n)^{1+\epsilon}$. Probably

(8) $$f_2(n)/(\log n)^k \to \infty$$

for every k which would make Grimm's conjecture likely in view of the fact that "probably"

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* K. Ramachandra just informed us that he can prove $f_2(n) > c \log n (\log \log n)^{1/4}$

(9) $$\lim\,(p_{r+1} - p_r) / (\log p_r)^k \to 0$$

for sufficiently large k. We certainly do not see how to prove (8) but this may be due to the fact that we overlook a simple idea. On the other hand the proof of (9) seems beyond human ingenuity at present.

In view of [2]

$$\underline{\lim}\ \frac{p_{r+1} - p_r}{\log p_r} < 1 .$$

Theorem 3 shows that Grimm's conjecture holds for infinitely many sets of composite numbers between consecutive primes.

THEOREM 4. For infinitely many n

$$f_5(n) > \exp(c_9\ (\log n \log \log n)^{1/2}).$$

A well known theorem of de Bruijn [1] implies that for an absolute constant c_9 the number of integers $m < n$ for which

(10) $$P(m) < \exp(c_9\ (\log n \log \log n)^{1/2})$$

is less than

(11) $$n \exp -(c_9\ (\log n \log \log n)^{1/2}).$$

(10) and (11) imply that there are $\exp(c_9\ (\log n \log \log n)^{1/2})$ consecutive integers not exceeding n all of whose greatest prime factors are greater than $\exp(c_9\ (\log n \log \log n)^{1/2})$, which proves Theorem 4.

It seems likely that for infinitely many n $f_3(n) < (\log n)^{c_{10}}$, but it is quite possible that for all n $f_3(n) > (\log n)^{c_{11}}$. We have no non-trivial upper bounds for $f_3(n)$, $f_4(n)$ or $f_5(n)$. It seems certain that $f_3(n) = o(n^{\epsilon})$ for every $\epsilon > 0$. It is difficult to guess good upper or lower bounds for $f_2(n)$.

Grimm observed that there are integers u and v, $u < v, P(u) = P(v)$ so that there is no prime between u and v e.g. $u = 24$, $v = 27$. It is easy to find many other such examples, but we cannot prove that there are infinitely many such pairs u_i, v_i and we cannot get good upper or lower bounds for $v_i - u_i$. Pólya's theorem of course implies $v_i - u_i \to \infty$.

It has been conjectured (at the present we cannot trace the conjecture) that if n_i and m_i have the same prime factors, then there is always a prime between n_i and m_i. We cannot get good upper or lower bounds on $m_i - n_i$.

Next we prove

THEOREM 5. Each of the inequalities

$$f_i(n) > f_{i+1}(n), \quad 0 \leq i \leq 4$$

have infinitely many solutions.

First we prove $f_o(n) > f_1(n)$ infinitely often. Put $n = pq$ where p and q are distinct primes, $q = (1 + o(1))\, p$, i.e. p and q are both of the form $(1 + o(1))\, n^{1/2}$. There is a largest k for which

$$f_o(pq - k) \geq k. \tag{12}$$

By theorem 1 none of the integers $pq - 1, \ldots, pq - k + 1$ can be multiples of p or q since $k = o(n^{1/2})$. Since k is maximal, by (12) the number of distinct prime factors of the product $(pq - k + 1) \ldots (pq)$ equals k. Thus the number of distinct prime factors of $(pq - k + 1) \ldots (pq - 1)$ is $k - 2$ hence $f_1(pq - k) < k - 1$ while $f_o(pq - k) \geq k$.

To prove $f_1(n) > f_2(n)$ infinitely often, observe that $f_1(pq - 1) > f_2(pq - 1)$ with p and q as above. Since $f_1(pq - 1) > \min(p, q)$, the primes p and q cannot both be used for f_2 but can be used for f_1.

Assume now $f_2(n) = k$ and assume that the set $n + 1, \ldots, n + k$ contains no power of a prime. Then $f_2(n) > f_3(n)$. Since $f_2(n) = k$ there must be r numbers $n + i_1, \ldots, n + i_r$ in the set which together with $n + k + 1$ are composed entirely of exactly r primes $q_1 < \ldots < q_r$ (we use Hall's theorem). Now none of these r numbers is a power of q_1 so their largest prime factors cannot all be distinct and thus $f_3(n) < k$.

Now clearly n^2 and $(n + 1)^2$ infinitely often have no power between them. This and the fact that $f_2(n^2) = o(n)$ gives infinitely often $f_2(n^2) > f_3(n^2)$. It might be interesting to try to determine the largest n such that $f_2(n) = f_3(n)$. We cannot even prove there is such an n.

Since $f_3(n)$ goes to infinity with n and $f_4(2^k - 3) = f_5(2^k - 3) = 2$, it is clear that $f_3(n) > f_4(n)$ infinitely often. Also $f_4(2^k - 1) > 2$ if $k > 1$ while $f_5(2^k - 1) = 2$. In fact it is easy to see that $f_4(2^k - 1)$ goes to infinity with k.

THEOREM 6. For all $n > n_o$, $f_1(n) > f_3(n)$.

Proof: Put $f_1(n) = k$. Then $(n + 1)\ldots(n + k)$ has exactly k distinct prime factors. If $f_3(n) = k$ then all these k primes must be the greatest prime factor of some $n + i$, $1 \leq i \leq k$. In particular 2 must be the greatest prime factor of $n + i$, $(n + i = 2^w)$ and similarly for 3 so that $n + i_2 = 2^v 3^w$.

Thus by theorem 1

$$(13) \qquad | 2^u - 2^v 3^w | < k < 2^{u/2} .$$

A well known theorem states that if $p_1, \ldots, p_r$ are r given primes and $a_1 < a_2 < \ldots$ is the set of integers composed of the p's then $a_{i+1} - a_i > a_i^{1-\epsilon}$ for every $\epsilon > 0$ and $i > i(\epsilon)$. This clearly contradicts (13), proving theorem 6.

It is not impossible that for every $n > n_0$

$$f_0(n) > f_1(n) > f_2(n) > f_3(n) > f_4(n)$$

but we are far from being able to prove this. It seems certain to us that $f_1(n) > f_2(n) > f_3(n)$ for all $n > n_0$ but we might hazard the guess that $f_0(n) = f_1(n)$ infinitely often, and perhaps $f_3(n) = f_4(n) = f_5(n)$ infinitely often. $f_4(2^k - 3) = f_5(2^k - 3) = 2$, thus $f_4(n) = f_5(n)$ has infinitely many solutions.

We can prove by using the methods of Theorem 4 that

$$f_3(n) < \exp\left((2 + o(1))(\log n \log\log n)^{1/2}\right)$$

for infinitely many n and that

$$f_2(n) < \exp\left(c \log n \log\log\log n / \log\log n\right)$$

for infinitely many n.

Perhaps our methods give that $f_0(n) < cn^{1/e}$ holds infinitely often and perhaps $f_0(n) < n^{\frac{1}{2}+\epsilon}$ holds for every $n > n_0$. All these and related questions we hope to investigate.

References

1. N.G. de Bruijn, On the number of positive integers ≤ X and free of prime factors > y. Indig. Math. 13 (1951) 50-60.

2. P. Erdös, The difference of consecutive prime numbers. Duke Math. J. 6 (1940), 438-441.

3. C. A. Grimm, A conjecture on consecutive composite numbers. Amer. Math. Monthly 76 (1969), 1126-1128.

4. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers. Illinois J. Math. 6 (1962), 64-94.

5. C. Siegel, Uber Naherungswerte Algebraischer Zahlen, Math. Annalen 84 (1921), 80-99, see also K. Mahler, Math. Annalen, 107 (1933), 691-730.