ON THE NUMBER OF POLES OF THE DYNAMICAL ZETA FUNCTIONS FOR BILLIARD FLOW

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Abstract. We study the number of the poles of the meromorphic continuation of the dynamical zeta functions η_N and η_D for several strictly convex disjoint obstacles satisfying non-eclipse condition. For η_N we obtain a strip $\{z \in \mathbb{C} : \operatorname{Re} s > \beta\}$ with infinite number of poles. For η_D we prove the same result assuming the boundary real analytic. Moreover, for η_N we obtain a characterisation of β by the pressure $P(2G)$ of some function G on the space Σ_A^f related to the dynamical characteristics of the obstacle.

1. Introduction

Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $r \geqslant 3$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with C^∞ smooth boundary and let $D = \bigcup_{j=1}^r D_j$. We assume that every D_j has non-empty interior and throughout this paper we suppose the following non-eclipse condition

$$D_k \cap \text{convex hull}\,(D_i \cup D_j) = \emptyset, \tag{1.1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition all periodic trajectories for the billiard flow in $\Omega = \mathbb{R}^d \setminus \mathring{D}$ are ordinary reflecting ones without tangential intersections to the boundary of D. We consider the (non-grazing) billiard flow φ_t (see Section 2 for the definition). Next the periodic trajectories will be called periodic rays. For any periodic ray γ, denote by $\tau(\gamma) > 0$ its period, by $\tau^\sharp(\gamma) > 0$ its primitive period, and by $m(\gamma)$ the number of reflections of γ at the obstacles. Denote by P_γ the associated linearized Poincaré map (see section 2.3 in [PS17] and Section 2 for the definition). Let $\mathcal{P}$ be the set of all oriented periodic rays. Notice that some periodic rays have only one orientation, while others admits two (see [CP22, §2.3] for more details). Let Π be the set of all primitive periodic rays. Then the counting function of the lengths of periodic rays satisfies

$$\sharp\{\gamma \in \Pi : \tau^\sharp(\gamma) \leqslant x\} \sim \frac{e^{hx}}{hx}, \quad x \to +\infty, \tag{1.2}$$

for some $h > 0$ (see for instance, [PP90, Theorem 6.5] for weakly mixing suspension symbolic flow and [Ika90a], [Mor91]). Hence there exists an infinite number of primitive periodic trajectories and applying (1.2), for every sufficiently small $\epsilon > 0$ one obtains the estimate

$$e^{(h-\epsilon)x} \leq \sharp\{\gamma \in \mathcal{P} : \tau(\gamma) \leqslant x\} \leqslant \mathrm{e}^{(h+\epsilon)x}, \ x \geq C_\epsilon \gg 1.$$

Moreover, for some positive constants c_1, C_1, f_1, f_2 we have (see for instance [Pet99, Appendix] and (A.1))

$$c_1 \mathrm{e}^{f_1 \tau(\gamma)} \leqslant |\det(\mathrm{Id} - P_\gamma)| \leqslant C_1 \mathrm{e}^{f_2 \tau(\gamma)}, \quad \gamma \in \mathcal{P}.$$

By using these estimates, define for $\mathrm{Re}(s) \gg 1$ two Dirichlet series

$$\eta_{\mathrm{N}}(s) = \sum_{\gamma \in \mathcal{P}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \quad \eta_{\mathrm{D}}(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},$$

where the sums run over all oriented periodic rays. The length $\tau^\sharp(\gamma)$, the period $\tau(\gamma)$ and $|\det(\mathrm{Id} - P_\gamma)|^{1/2}$ are independent of the orientation of γ. We consider also for $q \geq 1$, $q \in \mathbb{N}$, the zeta function

$$\eta_{\mathrm{q}}(s) = q \sum_{\gamma \in \mathcal{P}, m(\gamma) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \mathrm{Re}\, s \gg 1.$$

Clearly, $\eta_N(s) = \eta_1(s)$. These zeta functions are important for the analysis of the distribution of the scattering resonances related to the Laplacian in $\mathbb{R}^d \setminus \bar{D}$ with Dirichlet and Neumann boundary conditions on ∂D (see [CP22, §1] for more details).

It was proved in [CP22, Theorem 1.1 and Theorem 4.1] that η_q admit a meromorphic continuation to $\mathbb{C}$ with simple poles and integer residues. We have the equality

$$\eta_D(s) = \eta_2(s) - \eta_1(s), \ \mathrm{Re}\, s \gg 1, \tag{1.3}$$

hence η_D admits also a meromorphic continuation to $\mathbb{C}$ with simple poles and integer residues. The functions $\eta_q(s)$ are Dirichlet series with positive coefficients and by a classical theorem of Landau (see for instance, [Ber33, Théorème 1, Chapitre IV]) they have a pole $s = a_q$, where a_q is the abscissa of convergence of $\eta_q(s)$. On the other hand, from (1.3) it follows that some cancelations of poles are possible. In this direction, for $d = 2$ [Sto01] and for $d \geqslant 3$ under some conditions [Sto12] Stoyanov proved that there exists $\varepsilon > 0$ such that $\eta_{\mathrm{D}}(s)$ is analytic for $\mathrm{Re}\, s \geqslant a_1 - \varepsilon$. The same result has been proved for $d = 3$ and $a_1 > 0$ by Ikawa [Ika00].

The purpose of this paper is to prove that $\eta_q(s)$ has an infinite number of poles and to estimate $\beta \in \mathbb{R}$ such that the number of poles with $\mathrm{Re}\, s > \beta$ is infinite. The same questions are more difficult for $\eta_D(s)$ since the existence of at least one pole has been established only for obstacles with real analytic boundary [CP22, Theorem 1.3] and for obstacles with sufficiently small diameters [Ika90b], [Sto09]. Clearly, $a_q \le a_1$. We have $a_2 = a_1$, since if $a_2 < a_1$, the function η_D will have a singularity at a_1 which is impossible because η_D is analytic for $\operatorname{Re} s \ge a_1$ (see [Pet99, Theorem 1]).

Denote by $\operatorname{Res} \eta_q$, $\operatorname{Res} \eta_D$ the set of poles of η_q and η_D, respectively. For $\operatorname{Res} \eta_q$ we prove the following

Theorem 1.1. *For every $0 < \delta < 1$ there exists $\alpha_{\delta,q} < a_q$ such that for $\alpha < \alpha_{\delta,q}$ we have*

$$\sharp\{\mu_j \in \operatorname{Res} \eta_q : \operatorname{Re} \mu_j \ge \alpha, |\mu_j| \leqslant r\} \neq \mathcal{O}(r^\delta). \tag{1.4}$$

If the boundary ∂D is real analytic, the same result holds for $\operatorname{Res} \eta_D$.

More precisely, we show that for any $0 < \delta < 1$ there exists $\alpha_{\delta,q} < 0$ depending on the dynamical characteristics of D such that if $\alpha < \alpha_{\delta,q}$, for any constant $0 < C < \infty$ the estimate

$$\sharp\{\mu_j \in \operatorname{Res} \eta_q : \operatorname{Re} \mu_j \ge \alpha, |\mu_j| \leqslant r\} \leqslant C r^\delta, \quad r \geqslant 1$$

does not hold. Similar results have been proved for Pollicott-Ruelle resonances for Anosov flows [JZ17, Theorem 2], for Axiom A flows [JT23, Theorem 4.1] and for Neumann and Dirichlet scattering resonances for obstacles D satisfying (1.1) in [Pet02] and [CP22, Theorem 1.3], respectively. According to Theorem 1.1, it follows that for large $A > 0$ in the region $\mathcal{D}_A = \{z \in \mathbb{C} : \operatorname{Re} z > -A\}$ there are infinite number poles $\mu \in \operatorname{Res} \eta_1 \cap \mathcal{D}_A$ and infinite number poles $\nu \in \operatorname{Res} \eta_2 \cap \mathcal{D}_A$. Therefore if η_D is analytic in $\mathcal{D}_A$, by (1.3) we deduce that we must have an *infinite number of cancellations* of poles μ with poles ν and the corresponding residues of the cancelled poles μ and ν must coincide. For obstacles with real analytic boundary Theorem 1.1 shows that this is impossible.

Remark 1.1. The proof of Theorem 1.1 works if the function η_D is not entire. As we mentioned above, this holds for obstacles with real analytic boundary.

It is interesting to find the maximal number $\beta_q < a_q$ such that the strip $\{z \in \mathbb{C} : \operatorname{Re} z > \beta_q\}$ contains infinite number poles of η_q and to obtain so called *essential spectral gap*. This is a difficult open problem. Let $b_q < a_q$ be the abscissa of convergence of the series

$$\sum_{\gamma, m(\gamma) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma) e^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|}, \ \operatorname{Re} s \gg 1 \tag{1.5}$$

and let $\alpha = \max\{0, a_1\}$. In our second result we obtain a more precise result for $\operatorname{Res} \eta_1$.

Theorem 1.2. *For any small $\epsilon > 0$ we have*

$$\sharp\{\mu_j \in \mathrm{Res}\, \eta_1 : \ \mathrm{Re}\, \mu_j > (2d^2 + 2d - 1/2)(b_1 - 2\alpha) - \epsilon\} = \infty. \quad (1.6)$$

Notice that

$$2d^2 + 2d - 1/2 = 2(d^2 + d - 1) + 3/2 = 2 \dim G + 3/2,$$

where G is the $(d-1)$-Grassmannian bundle introduced in Section 2. In Appendix we prove that b_1 coincides with the abscissa of convergence of the series

$$\sum_{\gamma} \frac{\tau^\sharp(\gamma) e^{-s\tau(\gamma)}}{|\det(D_x \varphi_{\tau(\gamma)}|_{E_u(x)})|}, \ \mathrm{Re}\, s \gg 1, \quad (1.7)$$

where $E_u(x)$ is the unstable space of $x \in \gamma$ (see (2.1) for the notation). By using symbolic dynamics, we define (see (A.3)) a function $G(\xi, y) < 0$ on the space Σ_A^f related to dynamical characteristics of D (see Appendix) and prove the following

Proposition 1.2. *The abscissas of convergence a_1 and b_1 are given by*

$$a_1 = P(G), \ b_1 = P(2G), \quad (1.8)$$

$P(G)$ being the pressure of G defined by (A.2).

For $a_1 \leq 0$ we have $\alpha = 0$ and Theorem 1.2 is similar to [JZ17, Theorem 3] established for weakly mixing Anosov flows ψ_t, where instead of $b_1 = P(2G)$ one has the pressure $P(2\psi^u) < 0$ of the Sinai-Ruelle-Bowen potential

$$\psi^u(x) = -\frac{d}{dt}\Big(\log|\det D_x \psi_t|_{E_u(x)}|\Big)|_{t=0}.$$

Notice that for Anosov flow one has $P(\psi^u) = 0$, (see [BR75, Theorem 5]), while $a_1 = P(G)$ can be different from 0. More precise results for the poles of the semi-classical zeta function for contact Anosov flows have been obtained in [FT13], [FT17, Theorem 1.2].

Remark 1.3. The constant $2d^2 + 2d - 1/2$ in (1.6) is related to the estimate (3.3) of Fourier transform $\hat{F}_{A,1}$ in the local trace formula for $\eta_1(s)$ (see Theorem 3.2) and probably it is not optimal. A better estimate of $\hat{F}_{A,1}$ can be obtained if the bound of the number of poles (3.1) is improved (see for example, [AFW17], where the Hausdorff dimension of the trapped set K is involved).

We have $b_1 = b_2$ since the series

$$\sum_{\gamma} \frac{(-1)^{m(\gamma)} \tau^\sharp(\gamma) e^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|}, \ \mathrm{Re}\, s \gg 1 \quad (1.9)$$

is analytic for $\operatorname{Re} s \geq b_1$. We discuss this question at the end of Appendix. Theorem 1.2 can be generalized for $\operatorname{Res} \eta_2$ and one obtains (1.6). The proof works with some modifications.

The paper is organised as follows. In Section 2 we collect some definitions and notations from [CP22] which are necessary for the exposition. In particular, we define the non-grazing billiard flow φ_t, the $(d-1)$-Grassmannian bundle G, the bundles $\mathcal{E}_{k,\ell}$ over G and the operators $\mathbf{P}_{k,\ell}$, $0 \leq k \leq d$, $0 \leq \ell \leq d^2$. In Section 3 we obtain local trace formulas combining the results in [JT23, §6.1] and [CP22, Lemma 3.1]. In Section 4 we prove Theorems 1.1 and 1.2. Finally in Appendix we use symbolic dynamics and establish Proposition 1.2.

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2. Preliminaries

We recall the definition of billiard flow ϕ_t described in [CP22, §2.1]. Denote by $S\mathbb{R}^d$ the unit tangent bundle of $\mathbb{R}^d$ and by $\pi : S\mathbb{R}^d \to \mathbb{R}^d$ the natural projection. For $x \in \partial D_j$, denote by $n_j(x)$ the *inward unit normal vector* to ∂D_j at the point x pointing into D_j. Set

$$\mathcal{D} = \{(x, v) \in S\mathbb{R}^d \;:\; x \in \partial D\}.$$

We say that $(x, v) \in T_{\partial D_j}(\mathbb{R}^d)$ is incoming (resp. outgoing) if we have $\langle v, n_j(x)\rangle > 0$ (resp. $\langle v, n_j(x)\rangle < 0$). Introduce

$$\mathcal{D}_{\text{in}} = \{(x, v) \in \mathcal{D} \;:\; (x, v) \text{ is incoming}\},$$
$$\mathcal{D}_{\text{out}} = \{(x, v) \in \mathcal{D} \;:\; (x, v) \text{ is outgoing}\}.$$

Define the grazing set $\mathcal{D}_{\text{g}} = T(\partial D) \cap \mathcal{D}$ and obtain

$$\mathcal{D} = \mathcal{D}_{\text{g}} \sqcup \mathcal{D}_{\text{in}} \sqcup \mathcal{D}_{\text{out}}.$$

The billiard flow $(\phi_t)_{t\in\mathbb{R}}$ is the complete flow acting on $S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ which is defined as follows. For $(x, v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ set

$$\tau_{\pm}(x, v) = \pm \inf\{t \geqslant 0 : x \pm tv \in \partial D\}.$$

For $(x, v) \in \mathcal{D}_{\text{in/out}}$ denote by $v' \in \mathcal{D}_{\text{out/in}}$ the image of v by the reflexion with respect to $T_x(\partial D)$ at $x \in \partial D_j$, given by

$$v' = v - 2\langle v, n_j(x)\rangle n_j(x), \quad v \in S_x(\mathbb{R}^d), \quad x \in \partial D_j.$$

Then for $(x, v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\text{g}}$ define

$$\phi_t(x, v) = (x + tv, v), \quad t \in [\tau_-(x, v), \tau_+(x, v)],$$

while for $(x,v) \in \mathcal{D}_{\rm in/out}$, we set

$$\phi_t(x,v) = (x+tv, v) \quad \text{ if } \quad \begin{cases} \ (x,v) \in \mathcal{D}_{\rm out},\ t \in [0, \tau_+(x,v)]\,, \\ \text{or } (x,v) \in \mathcal{D}_{\rm in},\ t \in [\tau_-(x,v), 0]\,, \end{cases}$$

and

$$\phi_t(x,v) = (x+tv', v') \quad \text{ if } \quad \begin{cases} \ \ (x,v) \in \mathcal{D}_{\rm in},\ t \in]0, \tau_+(x,v)]\,, \\ \text{or } (x,v) \in \mathcal{D}_{\rm out},\ t \in [\tau_-(x,v'), 0[\,. \end{cases}$$

We extend ϕ_t to a complete flow still denoted by ϕ_t having the property

$$\phi_{t+s}(x,v) = \phi_t(\phi_s(x,v)), \quad t,s \in \mathbb{R}, \quad (x,v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D}).$$

Next we introduce the non-grazing set M as

$$M = B/\sim, \quad B = S\mathbb{R}^d \setminus \Big(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\rm g}\Big)\,,$$

where $(x,v) \sim (y,w)$ if and only if $(x,v) = (y,w)$ or

$$x = y \in \partial D \quad \text{ and } \quad w = v'.$$

The set M is endowed with the quotient topology. We change the notation and pass from ϕ_t to the non-grazing flow φ_t, which is defined on M as follows. For $(x,v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\rm in}$ define

$$\varphi_t([(x,v)]) = [\phi_t(x,v)], \quad t \in\,]\tau_-^{\rm g}(x,v), \tau_+^{\rm g}(x,v)[\,,$$

where $[z]$ denotes the equivalence class of $z \in B$ for the relation $\sim$, and

$$\tau_\pm^{\rm g}(x,v) = \pm \inf\{t > 0 : \phi_{\pm t}(x,v) \in \mathcal{D}_{\rm g}\}.$$

Thus φ_t is continuous, but the flow trajectory of (x,v) for times $t \notin\,]\tau_-^{\rm g}(x,v), \tau_+^{\rm g}(x,v)[$ is not defined. Clearly, we may have $\tau_\pm^{\rm g}(x,v) = \pm\infty$, while $\tau_\pm^{\rm g}(x,v) \neq 0$ for $(x,v) \in \mathcal{D}_{\rm in}$. Note that the above formula indeed defines a flow on M because each $(x,v) \in B$ has a unique representative in $(S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})) \cup \mathcal{D}_{\rm in}$.Following [DSW24, Section 3], we may define smooth charts on $M = B/\sim$ and φ_t becomes C^∞ non complete flow with respect to new charts.

Throughout we work with the smooth flow φ_t and denote by X its the generator. Let $A(z) = \{t \in \mathbb{R} \ :\ \pi(\varphi_t(z)) \in \partial D\}$. The *trapped set* K of φ_t is the set of points $z \in M$ which satisfy $-\tau_-^{\rm g}(z) = \tau_+^{\rm g}(z) = +\infty$ and

$$\sup A(z) = -\inf A(z) = +\infty.$$

By definition, $\varphi_t(z)$ is defined for all $t \in \mathbb{R}$ whenever $z \in K$. The flow φ_t is called *uniformly hyperbolic* on K, if for each $z \in K$ there exists a decomposition

$$T_z M = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{2.1}$$

which is $\mathrm{d}\varphi_t$-invariant with $\dim E_s(z) = \dim E_u(z) = d-1$, such that for some constants $C > 0$, $\nu > 0$, independent of $z \in K$, and some smooth norm $\|\cdot\|$ on TM, we have

$$\|\mathrm{d}\varphi_t(z)\cdot v\| \leqslant \begin{cases} C\mathrm{e}^{-\nu t}\|v\|, & v \in E_s(z), \quad t \geqslant 0, \\ C\mathrm{e}^{-\nu|t|}\|v\|, & v \in E_u(z), \quad t \leqslant 0. \end{cases} \tag{2.2}$$

The spaces $E_s(z)$ and $E_u(z)$ depend continuously on z (see [Has02, Section 2]).

The flow φ_t is uniformly hyperbolic on K (for the proof see [CP22, Appendix A]). Take a small neighborhood V of K in M, with smooth boundary and embed V into a compact manifold without boundary N. We extend arbitrarily X to obtain a smooth vector field on N, still denoted by X. The associated flow is still denoted by φ_t. Note that the new flow φ_t is now complete. Introducing the surjective map

$$\pi_M : B \ni (x,\xi) \to [(x,\xi)] \in M,$$

we have $\varphi_t \circ \pi_M = \pi_M \circ \phi_t$ and there is a bijection between periodic orbits of ϕ_t and φ_t presevering the periods of the closed trajectories of ϕ_t, while the corresponding Poincaré maps are conjugated (see [DSW24, Section 3]).

Consider the $(d-1)$-Grassmannian bundle

$$\pi_G : G \to N$$

over N. More precisely, for every $z \in N$, the set $\pi_G^{-1}(z)$ consists of all $(d-1)$-dimensional planes of $T_z N$. The dimension of $\pi_G^{-1}(z)$ is $d(d-1)$ and G is a smooth compact manifold with $\dim G = d^2 + d - 1$. We lift φ_t to a flow $\widetilde{\varphi}_t : G \to G$ defined by

$$\widetilde{\varphi}_t(z,E) = (\varphi_t(z), \mathrm{d}\varphi_t(z)(E)),\ z \in N,\ E \subset T_z N,\ \mathrm{d}\varphi_t(z)(E) \subset T_{\varphi_t(z)}N.$$

Introduce the set

$$\widetilde{K}_u = \{(z, E_u(z)) \ :\ z \in K\} \subset G.$$

Clearly, $\widetilde{K}_u$ is invariant under the action of $\widetilde{\varphi}_t$, since $\mathrm{d}\varphi_t(z)(E_u(z)) = E_u(\varphi_t(z))$. The set $\widetilde{K}_u$ will be seen as the trapped set of the restriction of $\widetilde{\varphi}_t$ to a neighborhood of $\widetilde{K}_u$ and the flow $\widetilde{\varphi}_t$ is uniformly hyperbolic on $\widetilde{K}$(see [BR75, Lemma A.3], [CP22, §2.5]). Let $\widetilde{X}$ be the generator of $\widetilde{\varphi}_t$ and let $\widetilde{V}_u$ be a small neighborhood of $\widetilde{K}_u$ in G with smooth boundary $\partial\widetilde{V}_u$ (see[CP22, §2.7]). Define

$$\Gamma_\pm(\widetilde{X}) = \{z \in \widetilde{V}_u :\ \widetilde{\varphi}_t(z) \in \widetilde{V}_u,\ \mp t > 0\}.$$

Denote by $\operatorname{clos}\widetilde{V}_u$ the closure of $\widetilde{V}_u$. Let $\tilde{\rho} \in C^\infty(\operatorname{clos}\widetilde{V}_u, \bar{\mathbb{R}}_+)$ be the defining function for $\widetilde{V}_u$ such that $\partial\widetilde{V}_u = \{z \in \operatorname{clos}\widetilde{V}_u : \tilde{\rho}(z) = 0\}$ and

$\mathrm{d}\tilde{\rho}(z) \neq 0$ for any $z \in \partial \widetilde{V}_u$. Following [GMT21, Lemma 2.3], for any small neighborhood $\widetilde{W}_0$ of $\partial \widetilde{V}_u$ there exists a vector field $\widetilde{Y}$ on clos $\widetilde{V}_u$ arbitrary close to $\widetilde{X}$ in C^∞-topology and flow $\widetilde{\psi}_t$ generated by $\widetilde{Y}$ with the properties:

$$(1)\ \ \mathrm{supp}\,(\widetilde{Y} - \widetilde{X}) \subset \widetilde{W}_0.$$

(2) (Convexity condition) For any defining function ρ of $\widetilde{V}_u$ and any $\omega \in \partial \widetilde{V}_u$ we have

$$\widetilde{Y}\rho(\omega) = 0 \Longrightarrow \widetilde{Y}^2 \rho(\omega) < 0.$$

(3) $\Gamma_\pm(\widetilde{X}) = \Gamma_\pm(\widetilde{Y})$, where $\Gamma_\pm(\widetilde{Y})$ is defined as above by $\widetilde{\psi}_t$.

By [DG16, Lemma 1.1], we may find a smooth extension of $\widetilde{Y}$ on G (still denoted by $\widetilde{Y}$) so that for every $\omega \in G$ and $T \geqslant 0$, we have

$$\omega, \widetilde{\psi}_T(\omega) \in \mathrm{clos}\ \widetilde{V}_u \quad \Longrightarrow \quad \widetilde{\psi}_t(\omega) \in \mathrm{clos}\ \widetilde{V}_u,\ \forall t \in [0, T]. \tag{2.3}$$

In the following we fix $\widetilde{V}_u, \widetilde{W}_0, \tilde{Y}$ and the flow $\tilde{\psi}_t$ with the properties mentioned above. Thus we obtain an *open hyperbolic system* satisfying the conditions (**A1**) – (**A4**) in [DG16, §0] (see also [JT23, §2.1]).

Next repeating the setup in [CP22, §2.6], we introduce some bundles passing to open hyperbolic system for bundles. First, define the tautological vector bundle $\mathcal{E} \to G$ by

$$\mathcal{E} = \{(\omega, u) \in \pi_G^*(TN)\ :\ \omega \in G,\ u \in [\omega]\},$$

where $[\omega] = E$ denotes the $(d-1)$ dimensional subspace of $T_{\pi_G(\omega)}N$ represented by $\omega = (z, E)$ and $\pi_G^*(TN)$ is the pullback bundle of TN. Second, introduce the "vertical bundle" $\mathcal{F} \to G$ by

$$\mathcal{F} = \{(\omega, W) \in TG\ :\ \mathrm{d}\pi_G(\omega) \cdot W = 0\},$$

which is a subbundle of the bundle $TG \to G$. The dimensions of the fibres $\mathcal{E}_\omega$ and $\mathcal{F}_\omega$ of $\mathcal{E}$ and $\mathcal{F}$ over ω are given by

$$\dim \mathcal{E}_\omega = d - 1, \quad \dim \mathcal{F}_\omega = \dim \ker \mathrm{d}\pi_G(\omega) = d^2 - d$$

for any $\omega \in G$ with $\pi_G(\omega) = z$. Finally, set

$$\mathcal{E}_{k,\ell} = \wedge^k \mathcal{E}^* \otimes \wedge^\ell \mathcal{F}, \quad 0 \leqslant k \leqslant d-1, \quad 0 \leqslant \ell \leqslant d^2 - d,$$

where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$, that is, we replace the fibre $\mathcal{E}_\omega$ by its dual space $\mathcal{E}^*_\omega$.

Next we use the notation $\omega = (z, \eta) \in G$ and $u \otimes v \in \mathcal{E}_{k,\ell}|_\omega$. By using the flow $\tilde{\psi}_t$, introduce a flow $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ by

$$\Phi_t^{k,\ell}(\omega, u\otimes v) = \Big(\tilde{\psi}_t(\omega),\ b_t(\omega)\cdot\Big[\big(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\big)^{\wedge k}(u) \otimes \mathrm{d}\widetilde{\psi}_t(\omega)^{\wedge\ell}(v)\Big]\Big), \tag{2.4}$$

with

$$b_t(\omega) = |\det \mathrm{d}\varphi_t(\pi_G(\omega))|_{[\omega]}|^{1/2} \cdot |\det\Big(\mathrm{d}\tilde{\psi}_t(\omega)|_{\ker \mathrm{d}\pi_G}\Big)|^{-1},$$

where ${}^{-\top}$ denotes the inverse transpose. Consider the transfer operator

$$\Phi_{-t}^{k,\ell,*} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$$

defined by

$$\Phi_{-t}^{k,\ell,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell}\big[\mathbf{u}(\widetilde{\psi}_{-t}(\omega))\big], \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}) \tag{2.5}$$

and let $\mathbf{P}_{k,\ell} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$ be the generator of $\Phi_{-t}^{k,\ell,*}$ defined by

$$\mathbf{P}_{k,\ell}\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_{-t}^{k,\ell,*}\mathbf{u}\Big)\Big|_{t=0}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

We obtain

$$\mathbf{P}_{k,\ell}(f\mathbf{u}) = (\mathbf{P}_{k,\ell}f)\mathbf{u} + f(\mathbf{P}_{k,\ell}\mathbf{u}),\ f \in C^\infty(G),\ \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

Notice that we obtain the same setup as in Definition 6.1 in [JT23, §6.1]. In the last paper the authors deal with a general Axiom A flow with several basic sets. In our case we have only one basic set and we may apply the results of [DG16] and [JT23]. With some constant $C > 0$ we have

$$\|e^{-t\mathbf{P}_{k,\ell}}\|_{L^2(G,\mathcal{E}_{k,\ell}) \to L^2(G,\mathcal{E}_{k,\ell})} \le Ce^{Ct},\ t \ge 0$$

and

$$(\mathbf{P}_{k,\ell} + s)^{-1} = \int_0^\infty e^{-t(\mathbf{P}_{k,\ell}+s)}dt : L^2(G, \mathcal{E}_{k,\ell}) \to L^2(G, \mathcal{E}_{k,\ell}),\ \mathrm{Re}\, s \gg 1.$$

Introduce the operator

$$\mathbf{R}_{k,\ell}(s) = \mathbf{1}_{\widetilde{V}_u}(\mathbf{P}_{k,\ell}+s)^{-1}\mathbf{1}_{\widetilde{V}_u} : C_c^\infty(\widetilde{V}_u, \mathcal{E}_{k,\ell}) \to \mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell}), \quad \mathrm{Re}(s) \gg 1,$$

where $\mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ denotes the space of $\mathcal{E}_{k,\ell}$-valued distributions. Applying [DG16, Theorem 1], we obtain a meromorphic extension of $\mathbf{R}_{k,\ell}(s)$ to the whole plane $\mathbb{C}$ with simple poles and positive integer residues.

For $\omega \in G$ and $t > 0$ consider the *parallel transport* map

$$\alpha_{\omega,t}^{k,\ell} = \alpha_{1,\omega,t} \otimes \alpha_{2,\omega,t} :\ \Lambda^k\mathcal{E}_\omega^* \otimes \Lambda^\ell\mathcal{F}_\omega \longrightarrow \Lambda^k\mathcal{E}^*_{\tilde{\psi}_t(\omega)} \otimes \Lambda^\ell\mathcal{F}_{\tilde{\psi}_t(\omega)}$$

given by

$$\mathbf{u}\otimes\mathbf{v}\longmapsto (e^{-t\mathbf{P}_{k,\ell}}(\mathbf{u}\otimes\mathbf{v}))(\widetilde{\psi}(t)),$$

where $\mathbf{u},\mathbf{v}$ are some sections of $\mathcal{E}_\omega^*$ and $\mathcal{F}_\omega$ over ω, respectively. The definition does not depend on the choice of $\mathbf{u}$ and $\mathbf{v}$ (see [DG16, Eq. (0.8)]). For a periodic trajectory $\widetilde{\gamma}: t\to\widetilde{\gamma}(t)=(\gamma(t),E_u(\gamma(t))$ with period T we define

$$\mathrm{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})=\mathrm{tr}\,\alpha_{\tilde{\gamma}(t),T}^{k,\ell}$$

(see [DG16], [CP22]) and the trace is independent of the choice of the point $\widetilde{\gamma}(t)\in\widetilde{\gamma}$.

Finally, if $\widetilde{\chi}\in C_c^\infty(\widetilde{V}_u)$ is equal to 1 near the trapping set $\widetilde{K}_u$ we have the Guillemin trace formula (see [DG16, (4.6)], [SWB23, §3.1],[CP22, §3.2]) with the flat trace

$$\mathrm{tr}^\flat(\widetilde{\chi}e^{-t\mathbf{P}_{k,\ell}}\widetilde{\chi})=\sum_{\widetilde{\gamma}}\frac{\tau^\sharp(\gamma)\,\mathrm{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})\delta(t-\tau(\gamma))}{|\det(\mathrm{Id}-\widetilde{P}_\gamma)|},\ t>0. \tag{2.6}$$

Here both sides are distributions on $(0,\infty)$ and the sum runs over all periodic orbits $\tilde{\gamma}$ of $\tilde{\varphi}_t$,

$$\widetilde{P}_\gamma=\mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})\big|_{\widetilde{E}_u(\omega_{\tilde{\gamma}})\oplus\widetilde{E}_s(\omega_{\tilde{\gamma}})}$$

is the linearized Poincaré map of the periodic orbit $\tilde{\gamma}(t)$ of the flow $\widetilde{\varphi}_t$ and $\omega_{\tilde{\gamma}}\in\mathrm{Im}(\tilde{\gamma})$ is any reference point taken in the image of $\tilde{\gamma}$.

To treat the zeta function related only to periodic rays with number of reflections $m(\gamma)\in q\mathbb{N}$, $q\geq 2$, we consider the setup introduced in [CP22, §4.1] and we recall it below. For $q\geqslant 2$ define the *q-reflection bundle* $\mathcal{R}_q\to M$ by

$$\mathcal{R}_q=\Big(\Big[S\mathbb{R}^d\setminus\Big(\pi^{-1}(\mathring{D})\cup\mathcal{D}_{\mathrm{g}}\Big)\Big]\times\mathbb{R}^q\Big)\Big/\approx, \tag{2.7}$$

where the equivalence classes of the relation $\approx$ are defined as follows. For $(x,v)\in S\mathbb{R}^d\setminus\Big(\pi^{-1}(\mathring{D})\cup\mathcal{D}_{\mathrm{g}}\Big)$ and $\xi\in\mathbb{R}^q$, we set

$$[(x,v,\xi)]=\{(x,v,\xi),(x,v',A(q)\cdot\xi)\}\qquad\text{if } (x,v)\in\mathcal{D}_{\mathrm{in}},\ (x,v')\in\mathcal{D}_{\mathrm{out}},$$

where $A(q)$ is the $q\times q$ matrix with entries in $\{0,1\}$ given by

$$A(q)=\begin{pmatrix}0 & & & 1\\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{pmatrix}.$$

Clearly, the matrix $A(q)$ yields a shift permutation

$$A(q)(\xi_1,\xi_2,...,\xi_q)=(\xi_q,\xi_1,...,\xi_{q-1}).$$

This indeed defines an equivalence relation since $(x, v') \in \mathcal{D}_{\text{out}}$ whenever $(x, v) \in \mathcal{D}_{\text{in}}$. Note that

$$A(q)^q = \mathrm{Id}, \quad \operatorname{tr} A(q)^j = 0, \quad j = 1, \dots, q-1. \tag{2.8}$$

Define a smooth structure of $\mathcal{R}_q$ as in [CP22, §4.1] and introduce the bundle

$$\mathcal{E}^q_{k,\ell} = \mathcal{E}_{k,\ell} \otimes \pi_G^* \mathcal{R}_q,$$

where $\pi_G^*\mathcal{R}_q$ is the pullback of $\mathcal{R}_q$ by π_G so $\pi_G^*\mathcal{R}_q \to G$ is a vector bundle over G. Consider a small smooth neighborhood V of K. We embed V into a smooth compact manifold without boundary N, and we fix an extension of $\mathcal{R}_q$ to N. Consider any connection ∇^q on the extension of $\mathcal{R}_q$ which coincides with d^q near K, and denote by

$$P_{q,t}(z) : \mathcal{R}_q(z) \to \mathcal{R}_q(\varphi_t(z))$$

the parallel transport of ∇^q along the curve $\{\varphi_\tau(z) \ : \ 0 \leqslant \tau \leqslant t\}$. We have a smooth action of φ_t^q on $\mathcal{R}_q$ which is given by the horizontal lift of φ_t

$$\varphi_t^q(z, \xi) = (\varphi_t(z), P_{q,t}(z) \cdot \xi), \quad (z, \xi) \in \mathcal{R}_q.$$

We may lift the flow φ_t to a flow $\Phi_t^{k,\ell,q}$ on $\mathcal{E}^q_{k,\ell}$ which is defined locally near $\widetilde{K}_u$ by

$$\begin{aligned}\Phi_t^{k,\ell,q}&(\omega, u \otimes v \otimes \xi) \\ &= \Big(\widetilde{\varphi}_t(\omega),\ b_t(\omega) \cdot \Big[\big(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\big)^{\wedge k}(u) \otimes (\mathrm{d}\widetilde{\varphi}_t(\omega))^{\wedge \ell}(v) \otimes P_{q,t}(z) \cdot \xi\Big]\Big)\end{aligned}$$

for any $\omega = (z, E) \in G$, $u \otimes v \otimes \xi \in \mathcal{E}^q_{k,\ell}(\omega)$ and $t \in \mathbb{R}$. Following [CP22, §4.1], we deduce that for any periodic orbit $\gamma = (\varphi_\tau(z))_{\tau \in [0, \tau(\gamma)]}$, the trace

$$\operatorname{tr}(P_{q,\gamma}) = \operatorname{tr}(P_{q,\varphi(z)}) = \begin{cases} q & \text{if} \quad m(\gamma) = 0 \mod q, \\ 0 & \text{if} \quad m(\gamma) \neq 0 \mod q \end{cases} \tag{2.9}$$

is independent of z. Define the transfer operator

$$\Phi_{-t}^{k,\ell,q,*} : C^\infty(G, \mathcal{E}^q_{k,\ell}) \to C^\infty(G, \mathcal{E}^q_{k,\ell})$$

by

$$\Phi_{-t}^{k,\ell,q,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell,q}[\mathbf{u}(\tilde{\varphi}_{-t}(\omega)],\ \mathbf{u} \in C^\infty(G, \mathcal{E}^q_{k,\ell})$$

and denote by $\mathbf{P}_{k,\ell,q}$ be the generator of $\Phi^{k,\ell,q,*}$. As above, we obtain the flat trace

$$\operatorname{tr}^\flat(\widetilde{\chi} e^{-t\mathbf{P}_{k,\ell,q}} \widetilde{\chi}) = q \sum_{\widetilde{\gamma}, m(\pi_G(\widetilde{\gamma})) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma) \operatorname{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell}) \delta(t - \tau(\gamma))}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|},\ t > 0. \tag{2.10}$$

We close this section by the following

Lemma 2.1 (Lemma 3.1, [CP22]). *For any periodic orbit $\tilde{\gamma}$ of the flow $\tilde{\varphi}_t$ related to a periodic orbit γ, we have*

$$\frac{1}{|\det(\mathrm{Id}-\widetilde{P}_\gamma)|}\sum_{k=0}^{d-1}\sum_{\ell=0}^{d^2-d}(-1)^{k+\ell}\,\mathrm{tr}(\alpha_{\tilde{\gamma}}^{k,\ell}) = |\det(\mathrm{Id}-P_\gamma)|^{-1/2}.$$

3. Local trace formula

In this section we apply the results of [DG16] and [JT23, §6.1] for vector bundles. For simplicity we will use the notations $\mathcal{E}_{k,\ell}=\mathcal{E}^1_{k,\ell}$, $\mathbf{P}_{k,\ell}=\mathbf{P}_{k,\ell,1}$, etc. For $\widetilde{\chi}\in C_c^\infty(\widetilde{V}_u)$ such that $\widetilde{\chi}\equiv 1$ near $\widetilde{K}_u$, by [DG16] and [JT23, §6.1] we conclude that for any integer $q\in\mathbb{N}$

$$\widetilde{\chi}(-i\mathbf{P}_{k,\ell,q}+s)^{-1}\widetilde{\chi}$$

has a meromorphic continuation to $\mathbb{C}$. Denote by $\mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q})$ the set of the poles of this continuation. Then for any constant $\beta>0$ it was proved in [JT23, (6.3)] that we have the upper bound

$$\sharp\mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q})\cap\{\lambda\in\mathbb{C},\ |\operatorname{Re}\lambda-E|\le 1,\ \operatorname{Im}\lambda\ge-\beta\}=\mathcal{O}(E^{d^2+d-1}). \tag{3.1}$$

In particular, there exists $C>0$ depending of β such that

$$\sharp\mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q})\cap\{\lambda\in\mathbb{C},\ |\lambda|\le E,\ \operatorname{Im}\lambda\ge-\beta\}\le CE^{d^2+d}+C.$$

Notice that the power d^2+d-1 comes from $\dim G$. Next for $\mathrm{Res}(-i\mathbf{P}_{k,\ell,q})$ we obtain as in [JT23] the following local trace formula.

Theorem 3.1 (Theorem 1.5 and (6.5), [JT23]). *For every $A>0$ and any $q\in\mathbb{N}$ there exists a distribution $F_A^{k,\ell,q}\in\mathcal{S}'(\mathbb{R})$ supported in $(0,\infty)$ such that*

$$\sum_{\mu\in\mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}),\operatorname{Im}\mu>-A} e^{-i\mu t}+F_A^{k,\ell,q}(t)$$
$$= q\sum_{\tilde{\gamma},\,m(\gamma)\in q\mathbb{N}}\frac{\tau^\sharp(\gamma)\,\mathrm{tr}(\alpha_{\tilde{\gamma}}^{k,\ell})\delta(t-\tau(\gamma))}{|\det(\mathrm{Id}-\widetilde{P}_\gamma)|},\ t>0. \tag{$3.2_{k,\ell,q}$}$$

Moreover for any $\epsilon>0$ the Fourier-Laplace transform $\hat{F}_A^{k,\ell,q}(\lambda)$ of $F_A^{k,\ell,q}(t)$ is holomorphic for $\operatorname{Im}\lambda<A-\epsilon$ and we have the estimate

$$|\hat{F}_A^{k,\ell,q}(\lambda)|=\mathcal{O}_{A,\epsilon,k,\ell,q}(1+|\lambda|)^{2d^2+2d-1+\epsilon},\ \operatorname{Im}\lambda<A-\epsilon. \tag{3.3}$$

Here $\gamma=\pi_G(\widetilde{\gamma})$.

As it was mentioned in [JT23, Section 6], the proof in [JT23, Section 4] with minor modifications works in the case of vector bundles. Combining the above result with Lemma 2.1, we obtain

Theorem 3.2. *For every* $A > 0$ *and any* $\epsilon > 0$ *there exists a distribution* $F_{A,q} \in \mathcal{S}'(\mathbb{R})$ *supported in* $(0, \infty)$ *with Fourier-Laplace transform* $\hat{F}_{A,q}(\lambda)$ *holomorphic for* $\operatorname{Im}\lambda < A - \epsilon$ *such that*

$$\sum_{k=0}^{d}\sum_{\ell=0}^{d^2-d}\sum_{\mu\in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}),\operatorname{Im}\mu>-A}(-1)^{k+\ell}e^{-i\mu t}+F_{A,q}(t) = q\sum_{\gamma,\, m(\gamma)\in q\mathbb{N}}\frac{\tau^\sharp(\gamma)\delta(t-\tau(\gamma))}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}},\ t>0, \tag{3.4_q}$$

where $\hat{F}_{A,q}(\lambda) = \sum_{k=0}^{d}\sum_{\ell=0}^{d^2-d}(-1)^{k+\ell}\hat{F}_A^{k,\ell,q}(\lambda)$ *satisfies the estimate* (3.3).

Choosing $q = 1$, we obtain a local trace formula for Neumann dynamical zeta function $\eta_N(s)$, introduced in Section 1. For the Dirichlet dynamical zeta function $\eta_D(s)$ given in Section 1 we use the representation (1.3) and applying $(3.4)_q$ with $q = 1, 2$, we obtain the local trace formula

$$\sum_{k=0}^{d}\sum_{\ell=0}^{d^2-d}\sum_{\mu\in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,2}),\operatorname{Im}\mu>-A}(-1)^{k+\ell}e^{-i\mu t} - \sum_{k=0}^{d}\sum_{\ell=0}^{d^2-d}\sum_{\mu\in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,1}),\operatorname{Im}\mu>-A}(-1)^{k+\ell}e^{-i\mu t}+F_{A,2}(t)-F_{A,1}(t) = \sum_{\gamma}\frac{(-1)^{m(\gamma)}\tau^\sharp(\gamma)\delta(t-\tau(\gamma))}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}},\ t>0. \tag{3.5}$$

Some resonances $\mu \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}),\ k+\ell$ odd, $q = 1, 2$ may cancel with some resonances $\nu \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}),\ k+\ell$ even, $q = 1, 2$ and appriori it is not clear if the meromorphic continuation of dynamical zeta functions $\eta_N(s)$ and $\eta_D(s)$ have infinite number poles. Notice that all poles are simples and the cancellations in $(3.4)_q$ could appear for terms with coefficients $+$ and $-$ related to $k+\ell$ even and $k+\ell$ odd, respectively. On the other hand, in (3.5) we have more possibilities for cancellations of poles.

4. Strip with infinite number poles

Proof of Theorem 1.1. We will prove Theorem 1.1 for η_D since the argument for η_q is completely similar and simpler. After cancelation all poles μ at the left hand side of (3.5) satisfy $\operatorname{Im}\mu \leq \alpha = \max\{0, a_1\}$. To avoid confusion, in the following we denote by $\widetilde{\mu}$ the poles μ in (3.5) which are not cancelled. Assume that for some $0 < \delta < 1$ and $0 \leq k \leq q,\ 0 \leq \ell \leq q^2 - q,\ q = 1, 2$ we have estimates

$$N_{A,k,\ell,q}(r) = \sharp\{\widetilde{\mu} \in \operatorname{Res}(-i\mathbf{P}_{k,\ell,q}) :\ |\widetilde{\mu}| \leq r,\ -A < \operatorname{Im}\widetilde{\mu} \leq \alpha\} \leq P(A, k, \ell, q, \delta) r^{\delta}. \quad (4.1)$$

We follow the argument in [JZ17, Section 5] and [CP22, Appendix B] with some modifications. Let $\rho \in C_0^{\infty}(\mathbb{R}; \mathbb{R}_+)$ be an even function with $\operatorname{supp}\rho \subset [-1, 1]$ such that

$$\rho(t) > 1 \quad \text{if} \quad |t| \leqslant 1/2,$$

and

$$\hat{\rho}(-\lambda) = \int \mathrm{e}^{it\lambda}\rho(t)\mathrm{d}t \geqslant 0, \quad k \in \mathbb{R}.$$

Let $(\ell_j)_{j\in\mathbb{N}}$ and $(m_j)_{j\in\mathbb{N}}$ be sequences of positive numbers such that $\ell_j \geqslant d_0 = \min_{k\neq m} \operatorname{dist}(D_k, D_m) > 0,\ m_j \geqslant \max\{1, \frac{1}{d_0}\}$ and let $\ell_j \to \infty,\ m_j \to \infty$ as $j \to \infty$. Set

$$\rho_j(t) = \rho(m_j(t - \ell_j)), \quad t \in \mathbb{R},$$

and introduce the distribution $\mathcal{F}_{\mathrm{D}} \in \mathcal{S}'(\mathbb{R}^+)$ by

$$\mathcal{F}_{\mathrm{D}}(t) = \sum_{\gamma\in\mathcal{P}} \frac{(-1)^{m(\gamma)}\tau^{\sharp}(\gamma)\delta(t - \tau(\gamma))}{|\det(I - P_\gamma)|^{1/2}}. \quad (4.2)$$

We have the following proposition established by Ikawa.

Proposition 4.1 (Prop. 2.3, [Ika90a])**.** *Suppose that the function* $s \mapsto \eta_{\mathrm{D}}(s)$ *cannot be prolonged as an entire function of* s*. Then there exists* $\alpha_0 > 0$ *such that for any* $\beta > \alpha_0$ *we can find sequences* $(\ell_j), (m_j)$ *with* $\ell_j \to \infty$ *as* $j \to \infty$ *such that for all* $j \geqslant 0$ *one has*

$$e^{\beta\ell_j} \leqslant m_j \leqslant \mathrm{e}^{2\beta\ell_j} \quad \textit{and} \quad |\langle\mathcal{F}_{\mathrm{D}}, \rho_j\rangle| \geqslant \mathrm{e}^{-\alpha_0\ell_j}.$$

We apply the local trace formula (3.5) to function $\rho_j(t)$. For $-A \leq \operatorname{Im}\zeta \leq \alpha$ we have

$$|\hat{\rho}_j(\zeta)| = m_j^{-1}|\hat{\rho}(m_j^{-1}\zeta)e^{-i\ell_j\zeta}| \leq C_N m_j^{-1} e^{\alpha\ell_j + m_j^{-1}\max(\alpha, A)}(1 + |m_j^{-1}\zeta|)^{-N}.$$

Then for $q = 1, 2$ and $-A \leq \operatorname{Im}\widetilde{\mu} \leq \alpha$ we obtain

$$\Big|\sum_{\operatorname{Im}\widetilde{\mu} > -A,\ \widetilde{\mu}\in\operatorname{Res}(-i\mathbf{P}_{k,\ell,q})} \langle e^{-i\widetilde{\mu}t}, \rho_j(t)\rangle\Big|$$

$$\leqslant C_{N,A} m_j^{-1} e^{\alpha \ell_j} \int_0^\infty (1+m_j^{-1} r)^{-N} dN_{A,k,\ell,q}(r)$$
$$= -C_{N,A} m_j^{-1} e^{\alpha \ell_j} \int_0^\infty \frac{d}{dr}\Big((1+m_j^{-1}r)^{-N}\Big) N_{A,k,l,q}(r) dr$$
$$\leqslant B_{N,A} P(A,k,\ell,q,\delta) m_j^{-(1-\delta)} e^{\alpha \ell_j} \int_0^\infty (1+y)^{-N-1} y^\delta dy$$
$$= A_N P(A,k,\ell,q,\delta) m_j^{-(1-\delta)} e^{\alpha\ell_j} \leq D_{A,k,\ell,q,\delta} e^{(-\beta(1-\delta)+\alpha)\ell_j}.$$

Next, applying (3.3), we have

$$\langle F_{A,q}, \rho_j\rangle = \int_{\mathbb{R}} \hat{F}_{A,q}(-\zeta)\hat{\rho}_j(\zeta) d\zeta = \int_{\mathbb{R}+i(\epsilon - A)} \hat{F}_{A,q}(-\zeta)\hat{\rho}_j(\zeta)d\zeta$$

and choosing $M = 2d^2 + 2d + 1$, we deduce

$$|\langle F_{A,q}, \rho_j\rangle| \leq C_{M,A,q} m_j^{-1} e^{(\epsilon - A)\ell_j} e^{m_j^{-1} \max\{A-\epsilon, \alpha\}}$$
$$\times \int (1+|\zeta|)^{2d^2+2d-1+\epsilon}(1+|m_j^{-1}\zeta|)^{-M} d\zeta$$
$$\leq D_{M,A,q} e^{(\epsilon - A)\ell_j} m_j^{2d^2+2d-1+\epsilon} \leq D_{M,A,q} e^{(\epsilon-A)\ell_j} e^{2(2d^2+2d-1+\epsilon)\beta\ell_j}.$$

According to [CP22, Theorem 1.3] for obstacles with real analytic boundary, the function η_D is not entire and we may apply Proposition 4.1. Taking together the above estimates and summing for $0 \leq k \leq d$, $0 \leq \ell \leq d^2 - d$ and $q = 1, 2$, we get

$$D_A e^{(-\beta(1-\delta)+\alpha)\ell_j} + E_A e^{(\epsilon - A)\ell_j} e^{2(2d^2+2d-1+\epsilon)\beta\ell_j} \geq e^{-\alpha_0 \ell_j}.$$

Here the constants D_A and E_A depend of A but they are independent of ℓ_j. We choose $\beta > \frac{\alpha_0+\alpha}{1-\delta}$. We fix β and $0 < \epsilon < 1$ and choose

$$A > 2(2d^2 + 2d - 1 + \epsilon)\beta + \epsilon + \alpha_0.$$

Fixing A, for $\ell_j \to \infty$ we obtain a contradiction. This completes the proof of Theorem 1.1 for η_D.

To deal with Res η_q, $q \geq 2$, we choose a periodic ray γ_0 with q reflections, $\ell_j = j\tau^\sharp(\gamma_0)$, $m_j = e^{\beta\ell_j}$ and apply the lower bound

$$\Big|\langle \sum_{\gamma,\, m(\gamma)\in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\delta(t-\tau(\gamma))}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \rho_j\rangle\Big| \geq c e^{-c_0\ell_j},\ \forall j \geq 0$$

with $c > 0, c_0 > 0$ independent of ℓ_j. For $q = 1$ we choose $\ell_j = j\tau^\sharp(\gamma)$, $m_j = e^{\beta\ell_j}$ with some periodic ray γ and obtain the above estimate. Repeating the argument for η_D, we prove (1.4). ■

Proof of Theorem 1.2. We follow the approach of F. Naud in [JZ17, Appendix A]. Let $0 \leq \rho \in C_0^\infty(-1,1)$ be the function introduced above. For $\xi \in \mathbb{R}$ and $t > \max\{d_0, 1\}$ introduce the function

$$\psi_{t,\xi}(s) = e^{i\xi t}\rho(s-t),\ \xi \in \mathbb{R}.$$

We apply the trace formula $(3.4)_1$ to $\psi_{t,\xi}$. As above denote by $\widetilde{\mu}$ the poles which are not cancelled in the left hand side of $(3.4)_1$. Assume that for $0 \leq k \leq d$ and $0 \leq \ell \leq d^2 - d$ we have

$$\sharp\{\widetilde{\mu} \in \mathrm{Res}\,(-i\mathbf{P}_{k,l}) :\ -A-\epsilon \leq \mathrm{Im}\,\widetilde{\mu} \leq \alpha\} = P(A,k,\ell,\epsilon) < \infty. \quad (4.3)$$

First, we have

$$|\hat{\psi}_{t,\xi}(\zeta)| \leq C_N e^{t\,\mathrm{Im}\,\zeta + |\,\mathrm{Im}\,\zeta|}(1 + |\,\mathrm{Re}\,\zeta - \xi|)^{-N}.$$

For $-A \leq \mathrm{Im}\,\widetilde{\mu} \leq \alpha$ and $N = 1$ the sum of terms involving poles $\widetilde{\mu}$ in $(3.4)_1$ can be bounded by $\frac{C_1 e^{\alpha t}}{1+|\xi|}$ with constant $C_1 > 0$ depending of $P(A,k,\ell,\epsilon)$ and $\exp(\max\{A,\alpha\})$. Second, by using (3.3) for $\hat{F}_{A,1}$, one deduces

$$|\langle F_{A,1}, \psi_{t,\xi}\rangle| \leq C_2 e^{(\epsilon - A)(t-1)}(1+|\xi|)^{2d^2+2d-1+\epsilon}.$$

Setting

$$S(t,\xi) = \sum_{\gamma} \frac{e^{i\xi\tau(\gamma)}\tau^\sharp(\gamma)\rho(\tau(\gamma)-t)}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},$$

we get

$$|S(t,\xi)| \leq \frac{C_1 e^{\alpha t}}{1+|\xi|} + C_A e^{-(A-\epsilon)t}(1+|\xi|)^{2d^2+2d-1+\epsilon}.$$

Now consider the Gaussian weight

$$G(t,\sigma) = \sigma^{1/2}\int_{\mathbb{R}} |S(t,\xi)|^2 e^{-\sigma\xi^2/2}d\xi,\ 0 < \sigma < 1.$$

The estimate for $|S(t,\xi)|$ yields

$$|S(t,\xi)|^2 \leq \frac{2C_1^2 e^{2\alpha t}}{(1+|\xi|)^2} + 2C_A^2 e^{-2(A-\epsilon)t}(1+|\xi|)^{2(2d^2+2d-1+\epsilon)}$$

and

$$G(t,\sigma) \leq C_1'\sigma^{1/2}e^{2\alpha t} + C_A'\sigma^{-(2d^2+2d-1+\epsilon)}e^{-2(A-\epsilon)t}. \quad (4.4)$$

On the other hand, taking into account only the terms with $\tau(\gamma) = \tau(\gamma')$, we get

$$G(t,\sigma) = \sqrt{2\pi}\sum_{\gamma}\sum_{\gamma'} \frac{\tau^\sharp(\gamma)\tau^\sharp(\gamma')e^{-(\tau(\gamma)-\tau(\gamma'))^2/2\sigma}\rho(\tau(\gamma)-t)\rho(\tau(\gamma')-t)}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}|\det(\mathrm{Id}-P_{\gamma'})|^{1/2}}$$
$$\geq c \sum_{t-1/2\leq\tau(\gamma)\leq t+1/2} \tau^\sharp(\gamma)|\det(\mathrm{Id}-P_\gamma)|^{-1} \tag{4.5}$$

with $c > 0$ independent of t and σ.

Set $\tau(\gamma) = T_\gamma$, $\tau^\sharp(\gamma) = T^\sharp_\gamma$, $a_\gamma = \frac{T^\sharp_\gamma}{|\det(\mathrm{Id}-P_\gamma)|}$. Recall that b_1 is the abscissa of convergence of Dirichlet series (1.5) with $q = 1$.

Case 1. $b_1 < 0$.
Let the lengths of the periodic rays be arranged as follows

$$T_1 \leq T_2 \leq ... \leq T_n \leq ....$$

It is well known (see for instance, [Cot17]) that

$$b_1 = \limsup_{n\to\infty} \frac{\log|\sum_{T_n\leq T_\gamma} a_\gamma|}{T_n}.$$

We fix a small $\epsilon > 0$ so that $-\delta = b_1 - 3\epsilon/2 < 0$.There exists an increasing sequence $n_1 < n_2 < ... < n_m < ...$ such that $\lim n_j = +\infty$ and

$$\frac{\log|\sum_{T_{n_j}\leq T_\gamma} a_\gamma|}{T_{n_j}} \geq b_1 - \epsilon. \tag{4.6}$$

Choose m large so that

$$1 > e^{-\delta T_{n_j}} + 2e^{-\frac{\epsilon}{2}T_{n_j}},\ \frac{1}{T_{n_j}} > e^{-\frac{\epsilon}{2}T_{n_j}} \text{ for } j \geq m.$$

Set $q_1 = T_{n_m}$ and write

$$\sum_{q_1\leq T_\gamma} a_\gamma = \sum_{k=1}^\infty \sum_{kq_1\leq T_\gamma<(k+1)q_1} a_\gamma.$$

Assume that we have the estimates

$$\sum_{kq_1\leq T_\gamma<(k+1)q_1} a_\gamma \leq e^{-\delta k q_1},\ \forall k \geq 1. \tag{4.7}$$

Then

$$\sum_{q_1\leq T_\gamma} a_\gamma \leq e^{-\delta q_1}\sum_{j=0}^\infty e^{-j\delta q_1} = e^{-\delta q_1}\frac{1}{1-e^{-\delta q_1}} < \frac{1}{2}e^{(-\delta+\epsilon/2)q_1}.$$

Since $-\delta + \epsilon/2 = b_1 - \epsilon$, we obtain a contradiction with (4.6) for T_{n_m}. Consequently, there exists at least one $k_1 \geq 1$ such that

$$\sum_{k_1 q_1 \leq T_\gamma < (k_1+1)q_1} a_\gamma > e^{-\delta k_1 q_1}. \tag{4.8}$$

The series $\sum_{T_\gamma \geq (k_1+1)q_1} a_\gamma e^{-\lambda T_\gamma}$ has the same abscissa of convergence b_1. We repeat the procedure choosing $q_2 > (k_1+1)q_1$, and obtain the existence of $k_2 \geq 1$ such that

$$\sum_{k_2 q_2 \leq T_\gamma < (k_2+1)q_2} a_\gamma > e^{-\delta k_2 q_2}. \tag{4.9}$$

By iteration we find two sequences $\{q_j\}$, $\{k_j\}$ such that

$$q_{j+1} > (k_j+1)q_j,\ k_j \geq 1,$$

and a sequence of disjoint intervals

$$[k_j q_j, (k_j+1)q_j],\ j = 1, 2, ...$$

so that

$$\sum_{k_j q_j \leq T_\gamma \leq (k_j+1)q_j} a_\gamma > e^{-\delta k_j q_j}. \tag{4.10}$$

The periods q_j may change applying the above procedure but for simplicity we use the same notation.

Next, suppose that

$$\sum_{k_j q_j + p \leq T_\gamma \leq k_j q_j + p + 1} a_\gamma < e^{-\delta k_j q_j}/q_j,\ p = 0, 1, ..., q_j - 1.$$

By using triangle inequality, we obtain a contradiction with (4.10). Thus for some $0 \leq p_j \leq q_j - 1$ we have

$$\sum_{k_j q_j + p_j \leq T_\gamma \leq k_j q_j + p_j + 1} a_\gamma \geq e^{-\delta k_j q_j}/q_j > e^{(-\delta - \epsilon/2)k_j q_j}.$$

Choosing $t_j = k_j q_j + p_j + 1/2$, we deduce

$$\sum_{t_j - 1/2 \leq T_\gamma \leq t_j + 1/2} a_\gamma \geq e^{(b_1 - 2\epsilon)t_j}.$$

Therefore from (4.4) and (4.5) with $t = t_j$ we obtain

$$c_1 \sigma^{1/2} e^{2\alpha t_j} + c_2 \sigma^{-(2d^2 + 2d - 1 + \epsilon)} e^{-2(A - \epsilon)t_j} \geq e^{(b_1 - 2\epsilon)t_j} \tag{4.11}$$

with constants $c_1, c_2 > 0$ independent of t_j. Now choose

$$\sigma = c_1^{-2} e^{2(b_1 - 3\epsilon - 2\alpha)t_j} < 1.$$

Since

$$b_1 - 3\epsilon - (b_1 - 2\epsilon) + 2\epsilon = \epsilon,$$

we have

$$e^{-\epsilon t_j} + c_3 e^{-2(2d^2+2d-1/2+\epsilon)(b_1-3\epsilon-2\alpha)t_j} e^{-2(A-(1/2)\epsilon)t_j} \geq 1.$$

Taking

$$A = -(2d^2+2d-1/2)(b_1-2\alpha) + 3\epsilon(2d^2+2d-\frac{b_1-2\alpha}{3}+\epsilon)$$

and letting $t_j \to +\infty$, we obtain a contradiction. Consequently, for some $0 \leq k_0 \leq 0$, $0 \leq \ell_0 \leq d^2-d$, setting $\tilde{\epsilon} = 3\epsilon(2d^2+2d-\frac{b_1-2\alpha}{3}+\epsilon)+\epsilon$, we have

$$\sharp\{\widetilde{\mu} \in \mathrm{Res}\,(-i\mathbf{P}_{k_0,l_0}) :\ \mathrm{Im}\,\widetilde{\mu} > (2d^2+2d-1/2)(b_1-2\alpha) - \tilde{\epsilon}\} = \infty.$$

This implies (1.6) with ϵ replaced by $\tilde{\epsilon}$, observing that the poles $\widetilde{\mu} \in \mathrm{Res}\,(-i\mathbf{P}_{k_0,\ell_0})$ coincide with the poles $\widetilde{\lambda}$ of the meromorphic continuation of $\eta_1(i\lambda)$.

Case 2. $b_1 > 0$.
For b_1 we have the representation

$$b_1 = \limsup_{n\to\infty} \frac{\log|\sum_{T_\gamma \leq T_n} a_\gamma|}{T_n}.$$

We fix a small $\epsilon > 0$ so that $b_1 - 2\epsilon > 0$. For every small $\epsilon_1 > 0$ the estimates for the numbers of periodic rays in Section 1 imply

$$e^{(h-\epsilon_1)T_n} \leq n \leq e^{(h+\epsilon_1)T_n},\ T_n \geq C_{\epsilon_1}. \tag{4.12}$$

We choose $\epsilon_1 = \frac{h}{4b_1}\epsilon$ and arrange

$$\begin{aligned} \frac{h-\epsilon_1}{h+\epsilon_1}(b_1 - \frac{3}{2}\epsilon) &= (b_1 - \frac{3}{2}\epsilon) - \frac{2\epsilon_1}{h+\epsilon_1}(b_1-\frac{3}{2}\epsilon) \\ &> (b_1-\frac{3}{2}\epsilon) - \frac{2\epsilon_1 b_1}{h} = b_1 - 2\epsilon. \end{aligned} \tag{4.13}$$

There exists an increasing sequence $n_1 < n_2 < ... < n_m < ...$ such that $\lim n_j = +\infty$ and

$$\frac{\log|\sum_{T_\gamma \leq T_{n_j}} a_\gamma|}{T_{n_j}} \geq b_1 - \epsilon. \tag{4.14}$$

Set $\log n_1 = p_1$ for T_{n_1} large using (4.12), we get

$$\sum_{T_\gamma \leq T_{n_1}} a_\gamma \leq \sum_{k=0}^{[\frac{p_1}{h-\epsilon_1}]} \sum_{k < T_\gamma \leq (k+1)} a_\gamma.$$

For simplicity of the notation set $d_1 = b_1 - \frac{3}{2}\epsilon > 0$. Assume that for $k = 0, ..., [\frac{p_1}{h-\epsilon_1}]$ we have

$$\sum_{k < T_\gamma \leq (k+1)} a_\gamma \leq e^{\frac{h-\epsilon_1}{h+\epsilon_1} d_1 k}.$$

This implies

$$\sum_{T_\gamma \leq T_{n_1}} a_\gamma \leq \sum_{k=0}^{[\frac{p_1}{h-\epsilon_1}]} e^{\frac{h-\epsilon_1}{h+\epsilon_1} d_1 k} = \frac{e^{\frac{h-\epsilon_1}{h+\epsilon_1} d_1 ([\frac{p_1}{h-\epsilon_1}]+1)} - 1}{e^{\frac{h-\epsilon_1}{h+\epsilon_1} d_1} - 1}.$$

Applying the inequality $e^x - 1 \geq x$ for $x \geq 0$ and exploiting (4.12), we deduce

$$\sum_{T_\gamma \leq T_{n_1}} a_\gamma < \frac{e^{\frac{h-\epsilon_1}{h+\epsilon_1} d_1 ([\frac{p_1}{h-\epsilon_1}]+1)}}{b_1 - 2\epsilon} < \frac{e^{d_1}}{b_1 - 2\epsilon} e^{\frac{\log n_1}{h+\epsilon_1} d_1}$$

$$\leq \frac{1}{2} e^{(d_1 + \epsilon/2) T_{n_1}} = \frac{1}{2} e^{(b_1 - \epsilon) T_{n_1}}$$

for large T_{n_1} depending of $\frac{e^{d_1}}{b_1 - 2\epsilon}$. We obtain a contradiction with (4.14). Taking into account (4.13) , for some $0 \leq k_1 \leq [\frac{p_1}{h-\epsilon_1}]$ we have

$$\sum_{k_1 < T_\gamma \leq k_1 + 1} a_\gamma \geq e^{(b_1 - 2\epsilon) k_1}.$$

Following this procedure, we construct a sequence of integers $\{k_j\}, k_{j+1} > k_j + 2$ satisfying

$$\sum_{k_j \leq T_\gamma \leq k_j + 1} a_\gamma \geq e^{(b_1 - 2\epsilon) k_j}.$$

We choose $t_j = k_j + 1/2$ and arrange

$$\sum_{t_j - 1/2 \leq T_\gamma \leq t_j + 1/2} a_\gamma \geq e^{(b_1 - 2\epsilon) t_j}.$$

Finally, we obtain (4.11) and a repetition of the argument in Case 1 implies (1.6).

Case 3. $b_1 = 0$.
For small $u > 0$ consider the Dirichlet series

$$\eta_u(s) = \sum_\gamma \frac{T_\gamma^\sharp e^{-(s+u)T_\gamma}}{|\det(\mathrm{Id} - P_\gamma)|} = \sum_\gamma a_\gamma e^{-uT_\gamma} e^{-sT_\gamma}.$$

This series has abscissa de convergence $-u < 0$ and we may apply the results of the Case 1. For a suitable sequence $t_j \to \infty$ depending of $-u$ we obtain the estimates

$$e^{-u(t_j-1/2)} \sum_{t_j-1/2\leq T_\gamma\leq t_j+1/2} a_\gamma \geq \sum_{t_j-1/2\leq T_\gamma\leq t_j+1/2} a_\gamma e^{-uT_\gamma} \geq e^{(-u-2\epsilon)t_j}.$$

Consequently,

$$\sum_{t_j-1/2\leq T_\gamma\leq t_j+1/2} a_\gamma \geq e^{-u/2}e^{-2\epsilon t_j} > e^{(-u/2-2\epsilon)t_j}.$$

These lower bounds are the estimates (4.11) with b_1 replaced by $-u/2$. The argument in the Case 1 implies

$$\sharp\{\mu_j \in \mathrm{Res}\eta_1 : \mathrm{Re}\,\mu_j > (2d^2+2d-1/2)(-2\alpha)-(\epsilon+(2d^2+2d-1/2)u/2)\} = \infty.$$

For small u we arrange $(2d^2+2d-1/2)u/2 < \epsilon$ and since ϵ is arbitrary, we obtain (1.6) with $b_1 = 0$. This completes the proof of Theorem 1.2. ■

Appendix

In this Appendix we prove Proposition 1.2. First,

$$\det(\mathrm{Id}-P_\gamma) = \det(\mathrm{Id}-D_x\varphi_{T_\gamma}|_{E_s(x)})\det(\mathrm{Id}-D_x\varphi_{T_\gamma}|_{E_s(x)})$$

$$= \det(D_x\varphi_{T_\gamma}|_{E_u(x)})\det(\mathrm{Id}-D_x\varphi_{T_\gamma}|_{E_s(x)})\det(D_x\varphi_{-T_\gamma}|_{E_u(x)}-\mathrm{Id}),\ x\in\gamma.$$

Consequently,

$$|\det(\mathrm{Id}-P_\gamma)|^{-1} = |\det D_x\varphi_{T_\gamma}|_{E_u(x)}|^{-1}$$

$$\times|\det(\mathrm{Id}-D_x\varphi_{T_\gamma}|_{E_s(x)})|^{-1}|\det(\mathrm{Id}-D_x\varphi_{-T_\gamma}|_{E_u(x)})|^{-1}.$$

For large T_γ we have

$$\|D_x\varphi_{T_\gamma}|_{E_s(x)}\| \leq Ce^{-\delta T_\gamma},\ \|D_x\varphi_{-T_\gamma}|_{E_u(x)}\| \leq Ce^{-\delta T_\gamma},\ \delta > 0, \forall T_\gamma$$

with constants $C > 0$, $\delta > 0$ independent of T_γ since the flow φ_t is uniformly hyperbolic (see [CP22, Appendix A]). Thus for large T_γ we obtain

$$c_1|\det D_x\varphi_{T_\gamma}|_{E_u(x)}|^{-1} \leq |\det(\mathrm{Id}-P_\gamma)|^{-1} \leq C_1|\det D_x\varphi_{T_\gamma}|_{E_u(x)}|^{-1} \tag{A.1}$$

with $0 < c_1 < C_1$ independent of T_γ. We have

$$\det D_x\varphi_{T_\gamma}|_{E_u(x)} = e^{d_\gamma},\ x\in\gamma$$

with

$$d_\gamma = \log\Big(\lambda_{1,\gamma}...\lambda_{d-1,\gamma}) > 0,$$

$\lambda_{j,\gamma}$ being the eigenvalues of $D_x\varphi_{T_\gamma}|_{E_u(x)}$ with modulus greater than 1. The above estimate shows that the abscissa of convergence of the series

$$\sum_\gamma T_\gamma^\sharp e^{-sT_\gamma+\delta_\gamma},\ \delta_\gamma=-d_\gamma,\ \mathrm{Re}\, s\gg 1$$

coincides with b_1.

Our purpose is to express b_1 by some dynamical characteristics related to symbolic dynamics for several disjoint strictly convex obstacles. To do this, we recall some well known results and we refer to [Ika88b], [Ika88a], [Ika90a], [PP90] for more details. Let $A(i,j)_{i,j=1,\dots,r}$ be a $r\times r$ matrix such that

$$A(i,j)=\begin{cases}1 & \text{if } i\neq j,\\ 0 & \text{if } i=j.\end{cases}$$

Introduce the space

$$\Sigma_A=\{\xi=\{\xi_i\}_{i=-\infty}^{\infty},\ \xi_i\in\{1,,\dots,r\},\ A(\xi_i,\xi_{i+1})=1,\ \forall i\in\mathbb{Z}\}.$$

$$\Sigma_A^+=\{\xi=\{\xi_i\}_{i=0}^{\infty},\ \xi_i\in\{1,,\dots,r\},\ A(\xi_i,\xi_{j+1})=1,\ \forall i\geq 0\}.$$

Given $0<\theta<1$, define a metric d_θ on Σ_A by $d_\theta(\xi,\eta)=0$ if $\xi=\eta$ and $d_\theta(\xi,\eta)=\theta^k$ if $\xi\neq\eta$ and k is the maximal integer such that $\xi_i=\eta_i$ for $|i|<k$. Similarly, we define a metric d_θ^+ on Σ_A^+. Following [PP90, Chapter 1], for a function $F:\Sigma_A\to\mathbb{C}$ define

$$\mathrm{var}_k F=\sup\{|F(\xi)-F(\eta)|:\ \xi_i=\eta_i,\ |i|<k\}$$

and for $G:\Sigma_A^+\to\mathbb{C}$ define

$$\mathrm{var}_k G=\sup\{|G(\xi)-G(\eta)|:\ \xi_i=\eta_i,\ 0\leq i<k\}$$

Let $F_\theta(\Sigma_A)$, $F_\theta(\Sigma_A^+)$ be the set of Lipschitz functions with respect to metrics d_θ, d_θ^+, respectively, with norm

$$|||f|||_\theta=\|f\|_\infty+\|f\|_\theta,\ \|f\|_\theta=\sup_{k\geq 0}\frac{\mathrm{var}_k f}{\theta^k}.$$

Let σ_A be shift on Σ_A and Σ_A^+ given by

$$(\sigma_A\xi)_i=\xi_{i+1},\ \forall i\in\mathbb{Z},\ (\sigma_A\xi)_i=\xi_{i+1},\ \forall i\geq 0,$$

respectively. For every $\xi\in\Sigma_A$ there exists a unique reflecting ray $\gamma(\xi)$ with successive reflections points on $\dots.\partial D_{j-1},\partial D_j,\partial D_{j+1},\dots$, where the order of reflections is determined by the sequence (ξ) (see [Ika88a]). If $(P_j(\xi))_{j=-\infty}^{\infty}$ are the reflexion points of $\gamma(\xi)$, we define the function

$$f(\xi)=\|P_0(\xi)-P_1(\xi)\|.$$

It was proved in [Ika88a, Section 3], [PS10, Section 3] that one can construct a sequence of phase functions $\{\varphi_{\xi,j}(x)\}_{j=-\infty}^{\infty}$, such that for each j the phase $\varphi_{\xi,j}$ is smooth in a neighborhood $\mathcal{U}_{\xi,j}$ of the segment

$[P_j(\xi), P_{j+1}(\xi)]$ in $\mathbb{R}^d \setminus \mathring{D}$ and

(i) $\|\nabla \varphi_{\xi,j}(x)\| = 1$ on $\mathcal{U}_{\xi,j}$,

(ii) $\nabla \varphi_{\xi,j}(P_j(\xi)) = \frac{P_{j+1}(\xi) - P_j(\xi)}{\|P_{j+1}(\xi) - P_j(\xi)\|}$,

(iii) $\varphi_{\xi,j} = \varphi_{\xi,j+1}$ on $\partial D_{j+1} \cap \mathcal{U}_{\xi,j} \cap \mathcal{U}_{\xi,j+1}$,

(iv) for each $x \in \mathcal{U}_{\xi,j}$ the surface $C_{\xi,j}(x) = \{y \in \mathcal{U}_{\xi,j} : \varphi_{\xi,j}(x) = \varphi_{\xi,j}(y)\}$ is strictly convex with respect to its normal fiels $\nabla \varphi_{\xi,j}$.

Denote by $\kappa_j(\xi)$, $j = 1, ..., d-1$, the principal curvatures at $P_0(\xi)$ of $C_{\xi,0}(x)$ and introduce

$$g(\xi) = -\log \prod_{j=1}^{d-1} (1 + f(\xi)\kappa_j(\xi)).$$

Then

$$\prod_{j=1}^{d-1} \lambda_{j,\gamma(\xi)} = \prod_{k=1}^{m(\gamma(\xi))} \prod_{j=1}^{d-1} (1 + f(\sigma_A^k \xi)\kappa_j(\sigma_A^k \xi)).$$

It follows form the exponential instability of the billiard ball map (see [Ika88a], [Sto99]) that $f(\xi), g(\xi)$ become functions in $F_\theta(\Sigma_A)$ with $0 < \theta < 1$ depending on the geometry of D. We define

$$S_n h(\xi) = h(\xi) + h(\sigma_A \xi) + ... + h(\sigma_A^{n-1}\xi)$$

and for a periodic ray $\gamma(\xi)$ we obtain

$$T_{\gamma(\xi)} = S_{m(\gamma(\xi))} f(\xi),\ \delta_{\gamma(\xi)} = S_{m(\gamma(\xi))} g(\xi).$$

Consider the zeta function

$$Z(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma_A^n \xi = \xi} e^{S_n(-sf(\xi) + g(\xi))} \Big),\ \operatorname{Re} s \gg 1$$

and observe that

$$-\frac{d}{ds} Z(s) = \sum_{\gamma} T_\gamma^\sharp e^{-sT_\gamma + \delta_\gamma}.$$

Next, it is well known (see for instance [PP90, Chapter 1]) that given $h \in F_\theta(\Sigma_A)$, there exist functions $\tilde{h}, \chi \in F_{\theta^{1/2}}(\Sigma_A)$ such that

$$h(\xi) = \tilde{h}(\xi) + \chi(\sigma_A \xi) - \chi(\xi)$$

and $\tilde{h}(\xi) \in F_{\theta^{1/2}}(\Sigma_A^+)$ depends only on the coordinates $(\xi_0, \xi_1, ...)$. We denote this property by $h \sim \tilde{h}$. Choose $\tilde{f} \sim f$, $\tilde{g} \sim g$ with $\tilde{f}, \tilde{g} \in$

$F_{\theta^{1/2}}(\Sigma_A^+)$ and write

$$Z(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{(\sigma_A^+)^n \xi = \xi} e^{S_n(-s\tilde{f}(\xi) + \tilde{g}(\xi))}\Big).$$

The pressure $P(F)$ of a function $F \in C(\Sigma_A)$ is defined by

$$P(F) = \sup_{\nu} \Big(h(\nu, \sigma_A) + \int_{\Sigma_A} F d\nu \Big),$$

where $h(\nu, \sigma_A)$ is the measure entropy of σ_A with respect to ν and the supremum is taken over all probability measures ν on Σ_A invariant with respect to σ_A.

Following [PP90, Chapter 6], consider the suspended flow $\sigma_t^f(\xi, s) = (\xi, s + t)$ on the space

$$\Sigma_A^f = \{(\xi, s) : \xi \in \Sigma_A,\ 0 \leq s \leq f(\xi)\}$$

with identification $(\xi, f(\xi)) \sim (\sigma_A(\xi), 0)$. For a function $G \in C(\Sigma_A^f)$ define the pressure

$$P(G) = \sup_{\nu_f} \{ h(\nu_f, \sigma_t^f) + \int_{\Sigma_A^f} G d\nu_f \}, \tag{A.2}$$

where $h(\nu_f, \sigma_t^f)$ is the measure entropy and the supremum is taken over all probability measures ν_f on Σ_A^f invariant with respect to σ_t^f. The suspended flow σ_t^f is weakly mixing, if there are not $t \in \mathbb{R} \setminus \{0\}$ with the property

$$\frac{t}{2\pi} f(\xi) \sim M(\xi),$$

where $M(\xi) \in C(\Sigma_A : \mathbb{Z})$ has only integer values. According to [Sto99, Lemma 5.2] and [Pet99, Lemma 1], the flow σ_t^f is *weakly mixing*

Applying the results of [PP90, Chapter 6], we deduce that the abscissa of convergence b_1 of $Z(s)$ is determined as the root of the equation $P(-s\tilde{f} + \tilde{g}) = P(-sf + g) = 0$ with respect to s. This root is unique since $s \to P(-sf + g)$ is decreasing. Introduce the function

$$G(\xi, y) = -\frac{1}{2} \sum_{j=1}^{d-1} \frac{\kappa_j(\xi)}{1 + \kappa_j(\xi) y}. \tag{A.3}$$

Clearly,

$$g(\xi) = 2 \int_0^{f(\xi)} G(\xi, y) dy.$$

Then [PP90, Proposition 6.1] says that $P(-b_1 f + g) = 0$ is equivalent to $b_1 = P(2G)$. With the same argument we show that $a_1 = P(G)$.

This completes the proof of Proposition 1.2. ■

It easy to find a relation between $P(2G)$ and $P(g)$. Repeating the argument of [PS10, Section 3], one obtains that there exist probability measures ν_g, ν_0 on Σ_A invariant with respect to σ_A such that

$$\frac{P(g)}{\int f(\xi) d\nu_g} \leq b_1 \leq \frac{P(g)}{\int f(\xi) d\nu_0}.$$

Consequently, b_1 has the same sign as $P(g)$.

We close this Appendix proving that $b_1 = b_2$. Consider the zeta function

$$Z_1(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{(\sigma_A^+)^n \xi = \xi} e^{S_n(-s\tilde{f}(\xi) + \tilde{g}(\xi) + i\pi)}\Big)$$

related to (1.9). Introduce the complex Ruelle operator

$$(\mathcal{L}_s u)(\xi) = \sum_{\sigma_A \eta = \xi} e^{(-s\tilde{f} + \tilde{g} + i\pi)(\eta)} u(\eta), \; u \in F_{\theta^{1/2}}(\Sigma_A^+).$$

Then for $s = b_1$ this operator has no eigenvalues 1 since this implies that the operator

$$(L_{b_1} u)(\xi) = \sum_{\sigma_A \eta = \xi} e^{(-b_1 \tilde{f} + \tilde{g})(\eta)} u(\eta)$$

has eigenvalue (-1). This is impossible because from $P(-b_1\tilde{f} + \tilde{g}) = 0$ one deduces that L_{b_1} has eigenvalue 1 and all other eigenvalues of L_{b_1} have modulus strictly less than 1 (see [PP90, Theorem 2.2]).This shows that the function $Z_1(s)$ is analytic for $s = b_1$, hence (1.9) has the same property. Finally, similarly to (1.3), we write the function (1.9) as a difference of two Dirichlet series with abscissas of convergence b_1 and b_2. Therefore the inequality $b_2 < b_1$ leads to contradiction.

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