DYNAMICAL ZETA FUNCTIONS FOR BILLIARDS

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Abstract. Let $D \subset \mathbb{R}^d$, $d \geqslant 2$, be the union of a finite collection of pairwise disjoint strictly convex compact obstacles. Let $\mu_j \in \mathbb{C}$, $\operatorname{Im} \mu_j > 0$ be the resonances of the Laplacian in the exterior of D with Neumann or Dirichlet boundary condition on ∂D. For d odd, $u(t) = \sum_j \mathrm{e}^{i|t|\mu_j}$ is a distribution in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ and the Laplace transforms of the leading singularities of $u(t)$ yield the dynamical zeta functions $\eta_{\mathrm{N}}, \eta_{\mathrm{D}}$ for Neumann and Dirichlet boundary conditions, respectively. These zeta functions play a crucial role in the analysis of the distribution of the resonances. Under a non-eclipse condition, for $d \geqslant 2$ we show that η_{N} and η_{D} admit a meromorphic continuation in the whole complex plane. In the particular case when the boundary ∂D is real analytic, by using a result of Fried [Fri95], we prove that the function η_{D} cannot be entire. Following Ikawa [Ika88b], this implies the existence of a strip $\{z \in \mathbb{C} :\ 0 < \operatorname{Im} z \leqslant \alpha\}$ containing an infinite number of resonances μ_j for the Dirichlet problem.

1. Introduction

Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $r \geqslant 3$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with smooth boundary and let $D = \bigcup_{j=1}^r D_j$. Throughout this paper we suppose the following non-eclipse condition

$$D_k \cap \text{convex hull}\,(D_i \cup D_j) = \emptyset, \tag{1.1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition all periodic rays for the billiard flow in $\mathbb{R}^d \setminus \mathring{D}$ are ordinary reflecting ones without tangent segments to the boundary of D. Notice that if (1.1) is not satisfied, for generic perturbations of ∂D all periodic reflecting rays in $\mathbb{R}^d \setminus \mathring{D}$ have no segments tangent to ∂D (see Theorem 6.3.1 in [PS17]). We consider the (non grazing) billiard flow $(\varphi_t)_{t\in\mathbb{R}}$ (see §2.2 for a precise definition). For any periodic γ, denote by P_γ its associated linearized Poincaré map and by $\tau(\gamma)$ its period. Let $\mathcal{P}$ be the set of all periodic rays. The counting function of the lengths of periodic rays satisfies the bound

$$\sharp\{\gamma \in \mathcal{P} :\ \tau(\gamma) \leqslant \tau\} \leqslant \mathrm{e}^{a\tau}, \quad \tau > 0,$$

for some $a > 0$. Moreover, for some constants $C, b_1, b_2 > 0$ we have (see for instance [Pet99])

$$C\mathrm{e}^{b_1\tau(\gamma)} \leqslant |\det(\mathrm{Id} - P_\gamma)| \leqslant \mathrm{e}^{b_2\tau(\gamma)}, \quad \gamma \in \mathcal{P}.$$

By using these estimates, for $\mathrm{Re}(s) \gg 1$ we define two Dirichlet series

$$\eta_{\mathrm{N}}(s) = \sum_{\gamma\in\mathcal{P}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \quad \eta_{\mathrm{D}}(s) = \sum_{\gamma\in\mathcal{P}} (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},$$

where for any periodic γ, we denoted by $\tau^\sharp(\gamma)$ its primitive period, and by $m(\gamma)$ the number of reflexions of γ on the obstacles. Here the sums run over all oriented periodic rays. Notice that some periodic rays have only one orientation, while others admits two ones (see §2.3). On the other hand, the length $\tau^\sharp(\gamma)$, the period $\tau(\gamma)$ and $|\det(\mathrm{Id} - P_\gamma)|^{1/2}$ are independent of the orientation of γ.

The series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$ are related to the resonances of the self-adjoint operators $-\Delta_{\mathrm{b}}$, $\mathrm{b} = \mathrm{N}, \mathrm{D}$, acting on domains $\mathcal{D}_{\mathrm{b}} \subset \mathcal{H} = L^2(\mathbb{R}^d \setminus D)$, with Neumann and Dirichlet boundary conditions on ∂D, respectively. To explain this relation, consider the resolvents

$$\mathcal{R}_{\mathrm{b}}(\mu) = \left(-\Delta_{\mathrm{b}} - \mu^2\right)^{-1},$$

which are analytic in $\{\mu \in \mathbb{C} : \operatorname{Im}\mu < 0\}$. Then $\mathcal{R}_{\mathrm{b}}(\mu) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{b,loc}}$ has a meromorphic continuation for $\mu \in \mathbb{C}$ if d is odd, and in the logarithmic covering of $\mathbb{C} \setminus \{0\}$ if d is even (see [LP89, Chapter 5] for d odd and [DZ19, Chapter 4]). The poles μ_j, $\operatorname{Im}\mu_j > 0$, of these continuations are called *resonances*. Introduce the distribution $u \in \mathcal{D}'(\mathbb{R})$ given by the trace

$$u(t) = 2\mathrm{tr}_{L^2(\mathbb{R}^d)}\left(\cos(t\sqrt{-\Delta_{\mathrm{b}}}) \oplus 0 - \cos(t\sqrt{-\Delta_0})\right),$$

where Δ_0 is the free Laplacian in $\mathbb{R}^d$ and $\cos(t\sqrt{-\Delta_{\mathrm{b}}}) \oplus 0$ acts as 0 on $L^2(D)$. Then for d odd, [Mel82] (see also [BGR82] for a slightly weaker result) proved that we have, in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$,

$$u(t) = \sum_j m(\mu_j) \mathrm{e}^{i|t|\mu_j},$$

where $m(\mu_j)$ is the multiplicity of μ_j. Here in the notations we omitted the dependence on the boundary conditions. The series above converge in the sense of distributions since we have the bound $\sharp\{\mu_j : |\mu_j| \leqslant r\} \leqslant Cr^d$ for all $r > 0$ (see Section 4.3 in [DZ19]). The reader may also consult [Zwo97] and [DZ19] for a proof treating the singularity of $u(t)$ at $t = 0$. For d even, the situation is more complicated since the resonances are defined in a logarithmic covering of $\mathbb{C} \setminus \{0\}$. Let $\Lambda = \mathbb{C} \setminus \mathrm{e}^{i\frac{\pi}{2}\mathbb{R}^+}$ and for $\rho > 0$ let

$$\Lambda_\rho = \{\mu \in \Lambda : |\operatorname{Im}\mu| \leqslant \rho|\operatorname{Re}\mu|, \ 0 < \arg\mu_j < \pi\}.$$

Choose a function ψ in $C_c^\infty(\mathbb{R}; [0,1])$ equal to 1 in a neighborhood of 0 and denote by $\sigma_{\mathrm{b}}(\lambda)$ the scattering phase related to $-\Delta_{\mathrm{b}}$ (see [Zwo98] for the notation). Following the work of Zworski (Theorem 1 in [Zwo98]), there exists a function $v_{\rho,\psi} \in C^\infty(\mathbb{R} \setminus \{0\})$ such that for even dimension d in the sense of distributions $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ one has

$$\begin{aligned} u(t) = \sum_{\mu_j \in \Lambda_\rho} & m(\mu_j)\mathrm{e}^{i\mu_j|t|} + m(0) \\ & + 2\int_0^\infty \psi(\lambda)\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}(\lambda)\cos(t\lambda)\mathrm{d}\lambda + v_{\rho,\psi}(t), \end{aligned} \tag{1.2}$$

where $m(0)$ is a constant. The reader may consult [Sjö97] for a local trace formula involving the resonances. Concerning the singularities of the distribution $u(t)$, from

[BGR82] it follows that

$$\text{sing supp}\, u \subset \{\pm\tau(\gamma) :\ \gamma \in \mathcal{P}\}.$$

Under the condition (1.1), every periodic trajectory γ is an ordinary reflecting ray and the leading singularity of $u(t)$ at $t = T$ was described by Guillemin and Melrose [GM79]. More precisely, the singularity related to T has the form

$$\sum_{\gamma\in\mathcal{P},\tau(\gamma)=T} (-1)^{m(\gamma)}\tau^\sharp(\gamma)|\det(\mathrm{Id} - P_\gamma)|^{-1/2}\delta(t - T) + L^1_{\mathrm{loc}}(\mathbb{R}) \tag{1.3}$$

(see for instance, Corollary 4.3.4 in [PS17]), where for the Neumann problem the factor $(-1)^{m(\gamma)}$ must be omitted. Taking the sum of the Laplace transforms of the leading singularities of $u(t)|_{\mathbb{R}^+}$ related to $\tau(\gamma), \gamma \in \mathcal{P}$, we obtain the Dirichlet series $\eta_{\mathrm{N}}(s), \eta_{\mathrm{D}}(s)$.

The analytic singularities of $\eta_{\mathrm{N}}(s)$ and $\eta_{\mathrm{D}}(s)$ are important for the analysis of the distribution of the resonances (see [Ika88b, Ika90a, Ika90b, Ika92, Sto09, Pet08] and the papers cited there). By using the Ruelle transfer operator and symbolic dynamics (see [Ika90a, Pet99, Sto09, Mor91]), a meromorphic continuation of $s \mapsto \eta_{\mathrm{N}}(s), \eta_{\mathrm{D}}(s)$ has been proved in a domain $s_0 - \epsilon \leqslant \mathrm{Re}\, s$ with a suitable $\epsilon > 0$, where s_0 is the abscissa of absolute convergence of the Dirichlet series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$. Recently, a meromorphic continuation on $\mathbb{C}$ of the series

$$\sum_{\gamma\in\mathcal{P}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|}, \quad \mathrm{Re}(s) \gg 1, \tag{1.4}$$

has been proved by Küster–Schütte–Weich [KSW21] (see also [BSW21, Theorem 4.4] for results concerning weighted zeta functions). On the other hand, a meromorphic continuation in the whole complex plane of the semi-classical zeta function for contact Anosov flows was established by Faure–Tsujii [FT17]. Their zeta function is similar to the function $\zeta_{\mathrm{N}}(s)$ defined in (1.5) below. The meromorphic continuation of the Ruelle zeta function $\prod_{\gamma\in\mathcal{P}}(1 - \mathrm{e}^{-s\tau(\gamma)})^{-1}$ for general Anosov flows was established by Giulietti–Liverani–Pollicott [GLP13] (see also the work of Dyatlov–Zworski [DZ16] for another proof based on microlocal analysis). In this paper the series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$ are simply called dynamical zeta functions following previous works [Pet99, Pet08] and we refer to the book of Baladi [Bal18] for more references concerning zeta functions for hyperbolic dynamical systems.

Our main result is the following

Theorem 1. *Let $d \geqslant 2$ and let the obstacles $D_j, j = 1, ..., r$, satisfy the condition (1.1). Then the series $\eta_{\mathrm{N}}(s)$ and $\eta_{\mathrm{D}}(s)$ admit a meromorphic continuation to the whole complex plane with simple poles and integer residues.*

One may also consider the zeta functions $\zeta_{\mathrm{b}}(s)$ associated to the boundary conditions $\mathrm{b} = \mathrm{D}, \mathrm{N}$, defined for $\mathrm{Re}\, s$ large enough by

$$\zeta_{\mathrm{b}}(s) = \exp\left(-\sum_{\gamma\in\mathcal{P}} (-1)^{m(\gamma)\varepsilon(\mathrm{b})} \frac{\mathrm{e}^{-s\tau(\gamma)}}{\mu(\gamma)|\det(\mathrm{Id} - P_\gamma)|^{1/2}}\right), \tag{1.5}$$

where $\varepsilon(\mathrm{D}) = 1$, $\varepsilon(\mathrm{N}) = 0$ and $\tau(\gamma) = \mu(\gamma)\tau^\sharp(\gamma)$; $\mu(\gamma) \in \mathbb{N}$ is the repetition number. Notice that we have

$$\frac{\zeta_{\mathrm{b}}'(s)}{\zeta_{\mathrm{b}}(s)} = \eta_{\mathrm{b}}(s), \quad \mathrm{b} = \mathrm{D}, \mathrm{N}, \quad \operatorname{Re} s \gg 1. \tag{1.6}$$

In particular, since by the above theorem $\eta_{\mathrm{b}}(s)$ has simple poles with integer residues, it follows by a classical argument of complex analysis that we have the following

Corollary 2. Under the assumptions of Theorem 1 for $\mathrm{b} = \mathrm{D}, \mathrm{N}$, the function $s \mapsto \zeta_{\mathrm{b}}(s)$ extends meromorphically to the whole complex plane.

In fact, we will prove a slightly more general result. For $q \in \mathbb{N}$, $q \geqslant 2$, consider the Dirichlet series

$$\eta_q(s) = \sum_{\gamma \in \mathcal{P},\, m(\gamma) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma) \mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \quad \operatorname{Re}(s) \gg 1,$$

where the sum runs over all periodic rays γ with $m(\gamma) \in q\mathbb{N}$. We will show that $\eta_q(s)$ admits a meromorphic continuation to the whole complex plane, with simple poles and residues valued in $\mathbb{Z}/q$ (see Theorem 4). In particular, considering the function $\zeta_q(s)$ defined by

$$\zeta_q(s) = \exp\left(- \sum_{\gamma \in \mathcal{P},\, m(\gamma) \in q\mathbb{N}} \frac{\mathrm{e}^{-s\tau(\gamma)}}{\mu(\gamma)|\det(\mathrm{Id} - P_\gamma)|^{1/2}} \right), \quad \operatorname{Re} s \gg 1,$$

one gets $q\zeta_q'/\zeta_q = q\eta_q$. Thus the function $s \mapsto \zeta_q(s)^q$ extends meromorphically to the whole complex plane since its logarithmic derivative is $q\eta_q$ and by Theorem 4 the function $q\eta_q$ has simple poles with integer residues. One reason for which it is interesting to study these functions is the relation

$$\eta_{\mathrm{D}}(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \log \frac{\zeta_2(s)^2}{\zeta_{\mathrm{N}}(s)} = 2\eta_2(s) - \eta_{\mathrm{N}}(s), \tag{1.7}$$

showing that $\eta_{\mathrm{D}}(s)$ for $\operatorname{Re} s \gg 1$ is expressed as the difference of two Dirichlet series with positive coefficients. In particular, to show that $\eta_{\mathrm{D}}(s)$ has a meromorphic extension to $\mathbb{C}$, it is sufficient to prove that both series $\eta_{\mathrm{N}}(s)$ and $\eta_2(s)$ have this property.

The distribution of the resonances μ_j depends on the geometry of the obstacles and for trapping obstacles it was conjectured that there exists $\alpha > 0$ such that, for d odd,

$$N_{0,\alpha} = \sharp\{\mu_j \in \mathbb{C} \;:\; 0 < \operatorname{Im} \mu_j \leqslant \alpha\} = \infty. \tag{1.8}$$

For d even we must count

$$N_{0,\alpha} = \sharp\{\mu_j \in \mathbb{C} \;:\; 0 < \operatorname{Im} \mu_j \leqslant \alpha,\ 0 < \arg z < \pi\} \tag{1.9}$$

since a meromorphic extension of $\mathcal{R}_{\mathrm{D}}(\mu)$ is possible on the Riemann logarithmic surface $\Lambda = \{-\infty < \arg z < +\infty\}$. This conjecture was introduced by Ikawa [Ika90a] for d odd and it is known as the modified Lax-Phillips conjecture (MLPC). In this direction, for d odd, Ikawa [Ika88b, Ika90a] proved that for strictly convex disjoint obstacles satisfying (1.1) the existence of at least one singularity of $\eta_{\mathrm{N}}(s)$ or $\eta_{\mathrm{D}}(s)$ implies the existence of

$\alpha > 0$ for which (1.8) holds for the Neumann or Dirichlet boundary problem. Notice that the value $\alpha > 0$ in [Ika90a] is related to the singularity of $\eta_{\mathrm{D}}(s)$ and to some dynamical characteristics. The proof in [Ika90a] can be modified to cover also the case d even, applying the trace formula of Zworski (1.2) (see Appendix B). The existence of a singularity of the dynamical zeta function trivially holds for the Neumann problem since $\eta_{\mathrm{N}}(s)$ is a Dirichlet series with positive coefficients, and by a classical result, $\eta_{\mathrm{N}}(s)$ must have a singularity at $s_0 \in \mathbb{R}$, where $\mathrm{Re}\, s = s_0$ is the line of absolute convergence of $\eta_{\mathrm{N}}(s)$. Moreover, for d odd (see [Pet02]) there are constants $c_0 > 0$, $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$ it holds

$$\sharp \left\{ \mu_j \in \mathbb{C} : \ 0 < \mathrm{Im}\, \mu_j \leqslant \frac{c_0}{\varepsilon}, \ |\mu_j| \leqslant r \right\} \geqslant C_\varepsilon r^{1-\varepsilon}.$$

The situation for the Dirichlet problem is more complicated since $\eta_{\mathrm{D}}(s)$ is analytic for $\mathrm{Re}\, s \geqslant s_0$, s_0 being the abscissa of absolute convergence [Pet99]. Moreover, for $d = 2$ [Sto01] and for $d \geqslant 3$ under some conditions [Sto12] Stoyanov proved that there exists $\varepsilon > 0$ such that $\eta_{\mathrm{D}}(s)$ is analytic for $\mathrm{Re}\, s \geqslant s_0 - \varepsilon$. The reason of this cancellation of singularities is related to the change of signs in the Dirichlet series defining $\eta_{\mathrm{D}}(s)$, as it is emphasized by the relation (1.7). Despite many works in the physical literature concerning the n-disk problem (see for example [CVW97, Wir99, LZ02, PWB$^+$12, BWP$^+$13] and the references cited there), a rigorous proof of the (MLPC) was established only for sufficiently small balls [Ika90b] and for obstacles with sufficiently small diameters [Sto09]. In this direction we prove the following

Theorem 3. *Under the assumptions of Theorem* 1*, if moreover the boundary ∂D is real analytic, then the function η_{D} has at least one pole and the* (MLPC) *is satisfied for the Dirichlet problem, that is, there exists $\alpha > 0$ such that* (1.8) *(resp.* (1.9)*) holds if d is odd (resp. d is even).*

Our paper relies heavily on the works [DG16, KSW21] and we provide specific references in the text. For convenience of the reader we explain briefly the general idea of the proofs of Theorems 1 and 3. First, in §2 we make some geometric preparations. The non-grazing billiard flow (φ_t) acts on $M = B/\sim$, where

$$B = S\mathbb{R}^d \setminus (\pi^{-1}(\mathring{D}) \cup \mathcal{D}_g),$$

$\pi : S\mathbb{R}^d \to \mathbb{R}^d$ is the natural projection, $\mathcal{D}_{\mathrm{g}} = \pi^{-1}(\partial D) \cap TD$ is the grazing part and $(x, v) \sim (y, w)$ if and only if $(x, v) = (y, w)$ or $x = y \in \partial D$ and w is equal to the reflected direction of v at $x \in \partial D$. By using this factorization, the flow (φ_t) becomes continuous in M. However, to apply the Dyatlov–Guillarmou theory [DG16] in order to study the spectral properties of (φ_t) which are related to the dynamical zeta functions, we need to work with a *smooth flow.* For this reason we use a special *smooth structure* near ∂D with smooth charts introduced in the recent work of Küster–Schütte–Weich [KSW21] (see §2.2). In this smooth model, the flow (φ_t) is smooth, and it is uniformly hyperbolic when restricted to the compact trapped set K of φ_t (see §2.4). The periodic points are dense in K and for any $z \in K$ the tangent space T_zM has the decomposition $T_zM = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z)$ with unstable and stable spaces

$E_u(z)$, $E_s(z)$, where X is the generator of φ_t. A meromorphic continuation of the cut-off resolvent $\chi(X+s)^{-1}\chi$ with $\chi \in C_c^\infty(M)$ supported near K has been established in [DG16] in a general setting. As in [DZ16] and [DG16], the estimates on the wavefront set of the resolvent $\chi(X+s)^{-1}\chi$ allow to define its flat trace which is related to the series (1.4). This implies a meromorphic continuation of this series in $\mathbb{C}$ (see [KSW21]).

To prove a meromorphic continuation of the series $\eta_{\mathrm{N}}(s)$ which involves factors $|\det(\mathrm{Id} - P_\gamma)|^{-1/2}$ instead of $|\det(\mathrm{Id} - P_\gamma)|^{-1}$, a natural approach would consist to study the Lie derivative $\mathcal{L}_X$ acting on sections of the unstable bundle $E_u(z)$ (see for example [FT17, pp. 6–8]). However, in general, $E_u(z)$ is not smooth with respect to z, but only Hölder continuous. Thus we are led to change the geometrical setting as in the work of Faure–Tsujii [FT17] (notice that the Grassmann bundle introduced below also appears in [BR75] and [GL08]). Consider the Grasmannian bundle $\pi_G : G \to V$ over a neighborhood V of K; for every $z \in V$ the fiber $\pi_G^{-1}(z)$ is formed by all $(d-1)$-dimensional planes of T_zV. Define the trapping set $\widetilde{K}_u = \{(z, E_u(z)) \,:\, z \in K\} \subset G$ and introduce the natural lifted smooth flow $(\widetilde{\varphi}_t)$ on G (see §2.5). Then according to [BR75, Lemma A.3], the set $\widetilde{K}_u$ is hyperbolic for $(\widetilde{\varphi}_t)$. We introduce the tautological bundle $\mathcal{E} \to G$ by setting

$$\mathcal{E} = \{(\omega, v) \in \pi_G^*(TV) \,:\, \omega \in G,\ v \in [\omega]\},$$

where $[\omega]$ denotes the subspace of $T_{\pi_G(\omega)}V$ that $\omega \in G$ represents, and $\pi_G^*(TV)$ is the pull-back of the tangent bundle $TV \to V$ by π_G. Next, we define the vector bundle $\mathcal{F} \to G$ by

$$\mathcal{F} = \{(\omega, W) \in TG \,:\, \mathrm{d}\pi_G(w) \cdot W = 0\}$$

which is the "vertical subbundle" of the bundle $TG \to G$. Finally, set

$$\mathcal{E}_{k,\ell} = \wedge^k \mathcal{E}^* \otimes \wedge^\ell \mathcal{F}, \quad 0 \leqslant k \leqslant d-1, \quad 0 \leqslant \ell \leqslant d^2 - d,$$

where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$. One defines a suitable flow $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ as well as a transfer operator (see §2.6 for the notations)

$$\Phi_{-t}^{k,\ell,*} u(\omega) = \Phi_t^{k,\ell}[\mathbf{u}(\tilde{\varphi}_{-t}(\omega)], \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

For a periodic orbit γ of φ_t, this geometrical setting allows to express the term $|\det(\mathrm{Id} - P_\gamma)|^{-1/2}$ as a finite sum involving the traces $\mathrm{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})$ related to the periodic orbit $\widetilde{\gamma} = \{(\gamma(t), E_u(\gamma(t)) \,:\, t \in [0, \tau(\gamma)]\}$ of the flow $(\widetilde{\varphi}_t)$ (see §3.2 for the notation $\alpha_{\widetilde{\gamma}}^{k,\ell}$ and Lemma 3.1). This crucial argument explains the introduction of the bundles $\mathcal{E}_{k,\ell}$ and the related geometrical technical complications. In this context we may apply the Dyatlov–Guillarmou theory (see Theorem 1 in [DG16]) for the generators

$$\mathbf{P}_{k,\ell}\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_{-t}^{k,\ell,*}\mathbf{u}\Big)\Big|_{t=0}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell})$$

of the transfer operators $\Phi_{-t}^{k,\ell,*}$ (in fact, by using smooth connexion, we introduce a new operator $\mathbf{Q}_{k,\ell}$ which coincide with $\mathbf{P}_{k,\ell}$ near $\tilde{K}_u$ (see §2.8)). This leads to a meromophic continuation of the the cut-off resolvent $\widetilde{\chi}(\mathbf{Q}_{k,\ell} + s)^{-1}\widetilde{\chi}$, where $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ is equal to 1 on $\widetilde{K}_u$ (see 2.8 for the notations). By applying the Guillemin flat trace formula [Gui77] (see [DZ16, Appendix B] and Section 3 in [BSW21]), concerning

$$\mathrm{tr}^\flat\Big(\int_0^\infty \varrho(t)\widetilde{\chi}(\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u})\widetilde{\chi}\mathrm{d}t\Big), \quad \varrho \in C_c^\infty(0,\infty),$$

we obtain the meromorphic continuation of η_{N}. Finally, the meromorphic continuation of η_q is obtained in a similar way, by considering in addition a certain *q-reflexion bundle* $\mathcal{R}_q \to G$ on which the flow $\widetilde{\varphi}_t$ can be lifted (see §4.1).

The strategy to prove Theorem 3 is the following. First, the representation (1.7) tells us that, if η_{D} can be extended to an entire function, then the function $\zeta_2^2/\zeta_{\mathrm{N}}$ has neither zeros nor poles on the whole complex plane. For obstacles with real analytic boundary we may use real analytic charts near ∂D to define a real analytic structure on M which makes (φ_t) a real analytic flow. In this setting we may apply a result of Fried [Fri95] to the non-grazing flow φ_t lifted to the Grassmannian bundle, and show that the entier functions ζ_2 and ζ_{N} have finite order. This crucial point implies that the meromorphic function $\zeta_2^2/\zeta_{\mathrm{N}}$ has also finite order. Finally, by using Hadamard's factorisation theorem, one concludes that we may write $\zeta_2(s)^2/\zeta_{\mathrm{N}}(s) = \mathrm{e}^{Q(s)}$ for some polynomial $Q(s)$. This leads to $\eta_{\mathrm{D}}(s) = -Q'(s)$ and we obtain a contradiction. Notice that this argument works as soon as the entier functions ζ_2 and ζ_{N} have finite order. The recent work of Bonthonneau–Jézéquel [BJ20] about Anosov flows suggests that this should be satisfied for obstacles with *Gevrey* regular boundary ∂D. In particular, the (MLPC) should be true for such obstacles. However in this paper we are not going to study this generalization.

The paper is organised as follows. In §2 one introduces the geometric setting of the billiard flow (φ_t) and its smooth model. We define the Grasmann extension G and the bundles $\mathcal{E}, \mathcal{F}$, $\mathcal{E}_{k,l} = \Lambda^k\mathcal{E}^\star \otimes \Lambda^\ell\mathcal{F}$ over G. Next, we discuss the setting for which we apply the Dyatlov-Guillarmou theory [DG16] for some first order operator $\mathbf{Q}_{k,\ell}$ leading to a meromorphic continuation of the cut-off resolvent $\mathbf{R}_{k,\ell}(s) = \tilde{\chi}(\mathbf{Q}_{k,\ell} + s)^{-1}\tilde{\chi}$. In §3 we treat the flat trace of the resolvent $\mathbf{R}_\varepsilon^{k,\ell}(s) = \mathrm{e}^{-\varepsilon(\mathbf{Q}_{k,\ell}+s)}\mathbf{R}_{k,\ell}(s)$, $\varepsilon > 0$, and we obtain a meromorphic continuation of η_N. In §4 we study the dynamical zeta functions $\eta_q(s)$ for particular rays γ having number of reflections $m(\gamma) \in q\mathbb{N}$, $q \geqslant 2$. Applying the result for $\eta_2(s)$, we deduce the meromorphic continuation of η_D. Finally, in §5 we treat the modified Lax-Phillips conjecture for obstacles with real analytic boundary. In Appendix A we present a proof for $d \geqslant 2$ of the uniform hyperbolicity of the flow φ_t in the Euclidean metric in $\mathbb{R}^d$, while in Appendix B we discuss the modifications of the proof of Theorem 2.1 in [Ika90a] for even dimensions.

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2. Geometrical setting

2.1. **The billiard flow.** Let $D_1, \ldots, D_r \subset \mathbb{R}^d$ be pairwise disjoint compact convex obstacles, satisfying the condition (1.1), where $r \geqslant 3$. We denote by $S\mathbb{R}^d$ the unit tangent bundle of $\mathbb{R}^d$ and by $\pi : S\mathbb{R}^d \to \mathbb{R}^d$ the natural projection. For $x \in \partial D_j$, we denote by $n_j(x)$ the *inward unit normal vector* to ∂D_j at the point x pointing into D_j. Set $D = \bigcup_{j=1}^r D_j$ and

$$\mathcal{D} = \{(x, v) \in S\mathbb{R}^d \ : \ x \in \partial D\}.$$

We will say that $(x, v) \in T_{\partial D_j}\mathbb{R}^d$ is incoming (resp. outgoing) if $\langle v, n_j(x)\rangle > 0$ (resp. $\langle v, n_j(x)\rangle < 0$), and introduce

$$\mathcal{D}_{\mathrm{in}} = \{(x, v) \in \mathcal{D} \ : \ (x, v) \text{ is incoming}\},$$
$$\mathcal{D}_{\mathrm{out}} = \{(x, v) \in \mathcal{D} \ : \ (x, v) \text{ is outgoing}\}.$$

We define the grazing set $\mathcal{D}_{\mathrm{g}} = T(\partial D) \cap \mathcal{D}$ and one gets

$$\mathcal{D} = \mathcal{D}_{\mathrm{g}} \sqcup \mathcal{D}_{\mathrm{in}} \sqcup \mathcal{D}_{\mathrm{out}}.$$

The billiard flow $(\phi_t)_{t\in\mathbb{R}}$ is the complete flow acting on $S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ which is defined as follows. For $(x, v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ we set

$$\tau_\pm(x, v) = \pm \inf\{t > 0 : x \pm tv \in \partial D\}$$

and for $(x, v) \in \mathcal{D}_{\mathrm{in/out/g}}$ we denote by $v' \in \mathcal{D}_{\mathrm{out/in/g}}$ the image of v by the reflexion with respect to $T_x(\partial D)$ at $x \in \partial D$, that is

$$v' = v - 2\langle v, n_j(x)\rangle n_j(x), \quad v \in S_x\mathbb{R}^d, \quad x \in \partial D_j$$

(see Figure 1). By convention, we have $\tau_\pm(x, v) = \pm\infty$, if the ray $x + \pm tv$ has no

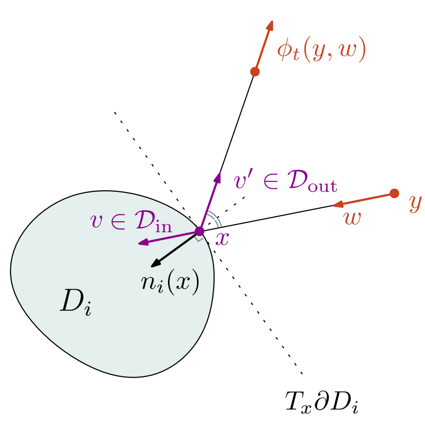

Figure 1. The billiard flow ϕ_t

common point with ∂D for $\pm t > 0$. Then for $(x, v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{g}}$ we define

$$\phi_t(x, v) = (x + tv, v), \quad t \in [\tau_-(x, v), \tau_+(x, v)],$$

while for $(x, v) \in \mathcal{D}_{\mathrm{in/out}}$, we set

$$\phi_t(x,v) = (x + tv, v) \quad \text{if} \quad \begin{cases} \quad (x,v) \in \mathcal{D}_{\mathrm{in}},\ t \in [0, \tau_+(x,v)]\,, \\ \text{or } (x,v) \in \mathcal{D}_{\mathrm{out}},\ t \in [\tau_-(x,v), 0]\,, \end{cases}$$

and

$$\phi_t(x,v) = (x + tv', v') \quad \text{if} \quad \begin{cases} \quad (x,v) \in \mathcal{D}_{\mathrm{out}},\ t \in \,]0, \tau_+(x,v)]\,, \\ \text{or } (x,v) \in \mathcal{D}_{\mathrm{in}},\ t \in [\tau_-(x,v), 0[\,. \end{cases}$$

Next we extend (ϕ_t) to a complete flow (which we still denote by (ϕ_t)) satisfying the property

$$\phi_{t+s}(x,v) = (\phi_t \circ \phi_s)(x,v), \quad t, s \in \mathbb{R}, \quad (x,v) \in S\mathbb{R}^d \setminus \pi^{-1}(D).$$

Strictly speaking, (ϕ_t) is not a flow, since the above flow property does not hold in full generality for $(x, v) \in \mathcal{D}_{\mathrm{in/out}}$. However, we can arrange it by considering an appropriate quotient space (see §2.2 below).

2.2. **A smooth model for the non-grazing billiard flow.** In this subsection, we briefly recall the construction of [KSW21, Section 3] which allows to obtain a smooth model for the non-grazing billiard flow. First, we define the non- grazing billiard table M as

$$M = B/\sim, \quad B = S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right),$$

where $(x, v) \sim (y, w)$ if and only if $(x, v) = (y, w)$ or

$$x = y \in \partial D \quad \text{and} \quad w = v'.$$

The set M is endowed with the quotient topology. We will change the notation and pass from ϕ_t to the non-grazing flow φ_t, which is defined on M as follows. For $(x, v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{in}}$ we define

$$\varphi_t([(x,v)]) = [\phi_t(x,v)], \quad t \in \,]\tau_-^{\mathrm{g}}(x,v), \tau_+^{\mathrm{g}}(x,v)[\,,$$

where $[z]$ denotes the equivalence class of the vector $z \in B$ for the relation $\sim$, and

$$\tau_\pm^{\mathrm{g}}(x,v) = \pm \sup\{t > 0 : \phi_{\pm t}(x,v) \in \mathcal{D}_{\mathrm{g}}\}.$$

Clearly, we may have $\tau_\pm^{\mathrm{g}}(x, v) = \pm\infty$. Note that this formula indeed defines a flow on M since each $(x, v) \in B$ has a unique representative in $(S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{in}}$. Thus (φ_t) is continuous but not complete and for times $t \notin \,]\tau_-^{\mathrm{g}}(x, v), \tau_+^{\mathrm{g}}(x, v)[\,$, the flow is not defined.

Following [KSW21, Section 3], we define smooth charts on $M = B/\sim$ as follows. Introduce the surjection map $\pi_M : B \to M$ by $\pi_M(x, v) = [(x, v)]$ and note that

$$\varphi_t \circ \pi_M = \pi_M \circ \phi_t. \tag{2.1}$$

Set $\mathring{B} := S\mathbb{R}^d \setminus \pi^{-1}(D)$. Then $\pi_M : \mathring{B} \to M$ is a homeomorphism onto its image $\mathcal{O}$. Let $\mathcal{G} = \pi_M(\mathcal{D}_{\mathrm{in}})$ be the gluing region. We consider the map $\pi_M^{-1} : \mathcal{O} \to \mathring{B}$ as a chart. Next we wish to define charts in an open neighborhood of $\mathcal{G}$. For every point $z_\star = (x_\star, v_\star) \in \mathcal{D}_{\mathrm{in}}$ let

$$F_{z_\star} : U_{z_\star} \times U_{z_\star} \to \mathcal{D}_{\mathrm{in}}$$

be a local smooth parametrization of $\mathcal{D}_{\rm in}$, where $U_{z_\star}$ is an open small neighborhood of 0 in $\mathbb{R}^{d-1}$. For small $\varepsilon_{z_\star} > 0$, we may define the map

$$\Psi_{z_\star} :]-\varepsilon_{z_\star}, \varepsilon_{z_\star}[\times U_{z_\star} \times U_{z_\star} \to M$$

by

$$\Psi_{z_\star}(t, y, w) = (\pi_M \circ \phi_t \circ F_{z_\star})(y, w). \tag{2.2}$$

Up to shrinking $U_{z_\star}$ and taking $\varepsilon_{z_\star}$ smaller, $\Psi_{z_\star}$ is a homeomorphism onto its image $\mathcal{O}_{z_\star} \subset M$, (see Corollary 4.3 in [KSW21]). Indeed, to see that $\Psi_{z_\star}$ is injective, let $F_{z_\star}(y_k, w_k) = (x_k, v_k) \in \mathcal{D}_{\rm in}$, $k = 1, 2$, and assume that $\pi_M \phi_{t_1}(x_1, v_1) = \pi_M \phi_{t_2}(x_2, v_2)$. Since the vectors in $\mathcal{D}_{\rm in}$ are transversal to ∂D, we see that for each $z \in \mathcal{O}_{z_\star}$, there is a unique $t \in]-\varepsilon_{z_\star}, \varepsilon_{z_\star}[$ such that $\varphi_t(z) \in \mathcal{G}$. In particular, we have $t_1 = 0$ if and only if $t_2 = 0$. In this case, $(x_1, v_1) = (x_2, v_2)$ since $\pi_M : \mathcal{D}_{\rm in} \to \mathcal{G}$ is injective. If $t_1 \neq 0, t_2 \neq 0$, then t_1 and t_2 have the same sign and by the infectivity of $\pi_M : \mathring{B} \to M$ and the definition of ϕ_t, we have

$$\begin{cases} (x_1 + t_1 v_1, v_1) = (x_2 + t_2 v_2, v_2) & \text{if} \quad t_1, t_2 > 0, \\ (x_1 + t_1 v_1', v_1') = (x_2 + t_2 v_2', v_2') & \text{if} \quad t_1, t_2 < 0, \end{cases}$$

where v_k' is the reflexion of v_k with respect to $T_{x_k}\partial D$ for $k = 1, 2$. Thus one concludes that $(t_1, x_1, v_1) = (t_2, x_2, v_2)$. As mentioned above, the directions in $\mathcal{D}_{\rm in}$ are transversal to the boundary ∂D. This implies that the maps $\Psi_{z_\star}$ are open ones. In particular, $\Psi_{z_\star}$ realises a homeomorphism onto its image $\mathcal{O}_{z_\star}$ and we declare the map $\Psi_{z_\star}^{-1} : \mathcal{O}_{z_\star} \to]-\varepsilon_{z_\star}, \varepsilon_{z_\star}[\times U_{z_\star} \times U_{z_\star}$ as a chart. Hence we obtain an open covering

$$\mathcal{G} \subset \bigcup_{z_\star \in \mathcal{D}_{\rm in}} \mathcal{O}_{z_\star}.$$

Note that if $\mathcal{O} \cap \mathcal{O}_{z_\star} \neq \emptyset$ for any $z_\star$, clearly the map

$$(t, x, v) \mapsto (\pi_M^{-1} \circ \Psi_{z_\star})(t, x, v) = (\phi_t \circ F_{z_\star})(x, v)$$

is smooth on $\Psi_{z_\star}^{-1}(\mathcal{O} \cap \mathcal{O}_{z_\star})$. On the other hand, assume that $\mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'} \neq \emptyset$ for some $z_\star, z_\star' \in \mathcal{D}_{\rm in}$. If $\pi_M(\phi_t(F_{z_\star}(x, v))) = \pi_M(\phi_s(F_{z_\star'}(y, w))) \in \mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'}$, then as above this yields $t = s$, $F_{z_\star}(x, v) = F_{z_\star'}(y, w)$, and we conclude that

$$\begin{aligned} (\Psi_{z_\star}^{-1} \circ \Psi_{z_\star'})(t, y, w) &= \left(\Psi_{z_\star}^{-1} \circ \pi_M \circ \phi_t \circ F_{z_\star'}\right)(y, w) \\ &= \left(\Psi_{z_\star}^{-1} \circ \pi_M \circ \phi_t \circ F_{z_\star}\right)\Big((F_{z_\star}^{-1} \circ F_{z_\star'})(y, w)\Big) \\ &= \left(t, (F_{z_\star}^{-1} \circ F_{z_\star})(y, w)\right). \end{aligned} \tag{2.3}$$

This shows that the change of coordinates $\Psi_{z_\star}^{-1} \circ \Psi_{z_\star'}$ is smooth on the set $\Psi_{z_\star'}^{-1}(\mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'})$, and these charts endow M with a smooth structure. It is easy to see that with this differential structure makes, the flow (φ_t) is smooth on M. Indeed, this is obvious far from the gluing region $\mathcal{G}$. Now let $z \in \mathcal{G}$ and $z_\star \in \mathcal{D}_{\rm in}$ be such that $\pi_M(z_\star) = z$. Then for $s, t \in \mathbb{R}$, with $|t| + |s|$ small, and $(y, w) \in U_{z_\star} \times U_{z_\star}$, we have

$$\begin{aligned} \left(\Psi_{z_\star}^{-1} \circ \varphi_s \circ \Psi_{z_\star}\right)(t, y, w) &= \left(\Psi_{z_\star}^{-1} \circ \varphi_s \circ \pi_M \circ \phi_t \circ F_{z_\star}\right)(y, w) \\ &= \left(\Psi_{z_\star}^{-1} \circ \pi_M \circ \phi_{t+s} \circ F_{z_\star}\right)(y, w) \\ &= (s + t, y, w). \end{aligned}$$

Consequently, the flow (φ_t) is also smooth near $\mathcal{G}$ and we obtain a smooth non-complete flow on M.

2.3. **Oriented periodic rays.** A periodic point of the billiard flow is a pair (x, v) lying in $S\mathbb{R}^d$, together with a number $\tau > 0$, such that $\phi_\tau(x, v) = (x, v)$. The number $\tau > 0$ is called the period of the periodic point (x, v). A *periodic trajectory* of (ϕ_t), or equivalently an *oriented periodic ray*, is by definition an equivalence class of periodic points, where we identify two periodic points (x, v) and (y, w), if they have same the period, and if there are $\tau_1, \tau_2 \in \mathbb{R}$ such that $\phi_{\tau_1}(x, v) = \phi_{\tau_2}(y, w)$. Of course, the map π_M induces a bijection between oriented periodic rays and periodic orbits of the non-grazing flow (φ_t). For each periodic orbit γ, we will denote by $\tau(\gamma)$ its period. Also, we will often identify a periodic orbit with a parametrization $\gamma : [0, \tau(\gamma)] \to S\mathbb{R}^d$.

Note that every oriented periodic ray is determined by a sequence

$$\alpha_\gamma = (i_1, \dots, i_k),$$

where $i_j \in \{1, \dots, r\}$, with $i_k \neq i_1$ and $i_j \neq i_{j+1}$ for $j = 1, ..., k-1$, such that γ has *successive reflections* on $\partial D_{i_1}, \dots, \partial D_{i_k}$. The sequence α_γ is well defined modulo cyclic permutation, and we say that the ray γ has type α_γ. The non-eclipse condition (1.1) implies that the reciprocal is true. More precisely, for any sequence $\alpha = (i_1, \dots, i_k)$ with $i_j \neq i_{j+1}$ for $j = 1, \dots, k-1$ and $i_k \neq i_1$, there exists a unique periodic ray γ such that $\alpha_\gamma = \alpha$ (see [PS17, Proposition 2.2.2 and Corollary 2.2.4]).

We conclude this paragraph by some remark on the oriented rays. For every oriented periodic ray γ generated by a periodic point (x, v) and period τ, one may consider the reversed ray $\bar{\gamma}$, generated by $(x, -v)$ and τ. There are two possibilities. For most rays, γ and $\bar{\gamma}$ give rise to different oriented periodic rays, even if theirs projections in $\mathbb{R}^d$ are the same. However it might happen that $\bar{\gamma}$ coincides with γ. This is the case, for example, if the ray γ has type $\alpha = (1, 2)$ (modulo permutation).

2.4. **Uniform hyperbolicity of the flow φ_t.** From now on, we will work exclusively with the flow (φ_t) defined on the smooth model described in §2.2. Let X be the generator of (φ_t). The trapped set K of (φ_t) is defined as the set of points $z \in M$ which satisfy $-\tau_-^{\mathrm{g}}(z) = \tau_+^{\mathrm{g}}(z) = +\infty$ and

$$\sup A(z) = -\inf A(z) = +\infty, \quad \text{where} \quad A(z) = \{t \in \mathbb{R} \ : \ \pi(\varphi_t(z)) \in \partial D\}.$$

By definition, $\varphi_t(z)$ is defined for all $t \in \mathbb{R}$ whenever $z \in K$. The flow (φ_t) is called uniformly hyperbolic on K, if for each $z \in K$ there exists a decomposition

$$T_z M = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{2.4}$$

which is $\mathrm{d}\varphi_t$-invariant (in the sense that $\mathrm{d}\varphi_t(E_\bullet(z)) = E_\bullet(\varphi_t(z))$ for $\bullet = u, s$), with $\dim E_s(z) = \dim E_u(z) = d-1$, such that for some constants $C > 0, \nu > 0$, independent of $z \in K$, and some smooth norm $\|\cdot\|$ on TM, we have

$$\|\mathrm{d}\varphi_t(z) \cdot v\| \leqslant \begin{cases} C\mathrm{e}^{-\nu t}\|v\|, & v \in E_s(z), \quad t \geqslant 0, \\ C\mathrm{e}^{-\nu|t|}\|v\|, & v \in E_u(z), \quad t \leqslant 0. \end{cases} \tag{2.5}$$

The spaces $E_s(z)$ and $E_u(z)$ depend continuously on z.

We may define the trapping set K_e for the flow (ϕ_t) in the Euclidean metric. Then $K = \pi_M(K_e)$. The uniform hyperbolicity on K_e of the flow (ϕ_t) in the Euclidean metric can be defined by the splitting of the tangent space $T_z(\mathbb{R}^d)$ for $z \in \mathring{B} \cap K_e$ (see Definition 2.10 in [KSW21] and Appendix A). Following this definition, one avoids the points $(x, v) \in K_e \cap \mathcal{D}_{\text{in}}$. Denote $\overline{\mathcal{D}}_{\text{in}} = \{(x, v) : x \in \partial D, |v| = 1, \langle v, n(x)\rangle \geqslant 0\}$ and define the *billiard ball map*

$$\mathbf{B} : \overline{\mathcal{D}}_{\text{in}} \ni (x, v) \longmapsto (y, R_y w) \in \overline{\mathcal{D}}_{\text{in}},$$

where $R_y : S_y\mathbb{R}^d \to S_y\mathbb{R}^d$ is the reflexion with respect to $T_y\partial D$ and

$$(y, w) = \phi_{\tau_+(x,v)}(x, v),\ \tau_+(x, v) = \inf\{t > 0 \ : \ \pi(\phi_t(x, v)) \in \partial D\}.$$

This map $\mathbf{B}$ is called *collision map* in [CM06], it is well defined near $K_e \cap \mathcal{D}_{\text{in}}$ and it is smooth (see for instance, [Kov88]). For $(x, v) \in K_e \cap \mathcal{D}_{\text{in}}$ we can write $\mathrm{d}\phi_t(x, v)$ as a composition of the differentials of smooth maps (see [CM06, §4.4] and Appendix A) and this is useful for the estimates of $\|\mathrm{d}\phi_t(x, v)\|$.

The uniform hyperbolicity of (ϕ_t) in the Euclidean metric implies the uniform hyperbolicity of (φ_t) in the smooth model (see [KSW21, Proposition 3.8]). Thus, to obtain (2.5), we may apply the uniform hyperbolcity of (ϕ_t) in the Euclidean metric on $\mathring{B} \cap K_e$ established for $d = 2$ in [Mor91] and [CM06, §4.4]. For $d \geqslant 3$, the same could perhaps be obtained by applying the results in [BCST03, §4]. The hyperbolicity at the points $z = (x, v) \in K_e$ which are not periodic must be justified and the stable/unstable spaces $E_s(z)/E_u(z)$ must be well determined; for $d \geqslant 3$ this seems to be not sufficiently detailed in the literature. Since the hyperbolicity of (φ_t) is crucial for our exposition, and for the sake of completeness, we present in Appendix A a proof of the uniform hyperbolicity as well as a construction of $E_s(z)$ and $E_u(z)$ for all $z \in \mathring{B} \cap K_e$.

2.5. **The Grassmann extension.** In what follows, we will take a small neighborhood V of K, with smooth boundary. We embed V into a compact manifold without boundary N. For example, we may take the double manifold N of the closure of V. This means that $N = \bar{V} \times \{0, 1\}/\sim$ and $(x, 0) \sim (x, 1)$ for all $x \in \partial V$. We arbitrarily extend X to obtain a smooth vector field on N, which we still denote by X. The associated flow is still denoted by (φ_t) (however note that this new flow (φ_t) is now complete).

For our exposition is important to introduce the $(d-1)$-Grassmann bundle

$$\pi_G : G \to N$$

over N. More precisely, for every $z \in N$, the set $\pi_G^{-1}(z)$ consists of all $(d-1)$-dimensional planes of T_zN. Moreover, $\pi_G^{-1}(z)$ can be identified with the Grasmannian $G_{d-1}(\mathbb{R}^{2d-1})$ which is isomorphic to $O(2d-1)/(O(d-1) \times O(d))$, $O(k)$ being the space of $(k \times k)$ orthogonal matrices with elements in $\mathbb{R}$. The dimension of $O(k)$ is $k(k-1)/2$, hence the dimension of $\pi_G^{-1}(z)$ is $d(d-1)$. Note that G is a smooth compact manifold. We may lift the flow φ_t to a flow $\widetilde{\varphi}_t : G \to G$ which is simply defined by

$$\widetilde{\varphi}_t(z, E) = (\varphi_t(z), \mathrm{d}\varphi_t(z)(E)), \quad z \in N, \quad E \subset T_zN, \quad \mathrm{d}\varphi_t(z)(E) \subset T_{\varphi_t(z)}N. \tag{2.6}$$

Introduce the set
$$\widetilde{K}_u := \{(z, E_u(z)) \;:\; z \in K\} \subset G.$$
Clearly, $\widetilde{K}_u$ is invariant under the action of $\widetilde{\varphi}_t$, since $\mathrm{d}\varphi_t(z)(E_u(z)) = E_u(\varphi_t(z))$. As K is a hyperbolic set, it follows from [BR75, Lemma A.3] that the set $\widetilde{K}_u$ will be hyperbolic for $\widetilde{\varphi}_t$ and we have a decomposition
$$T_\omega G = \mathbb{R}\widetilde{X}(\omega) \oplus \widetilde{E}_u(\omega) \oplus \widetilde{E}_s(\omega), \quad \omega \in \widetilde{K}_u.$$
Here $\widetilde{X}$ is the generator of the flow $(\widetilde{\varphi}_t)$ and the spaces $\widetilde{E}_s(\omega)$ and $\widetilde{E}_u(\omega)$ are defined as follows. For small $\varepsilon > 0$, let
$$W_s(z,\varepsilon) = \{z' \in M : \mathrm{dist}\,(\varphi_t(z), \varphi_t(z')) \leqslant \varepsilon \text{ for every } t \geqslant 0\}$$
and
$$W_u(z,\varepsilon) = \{z' \in M : \mathrm{dist}\,(\varphi_{-t}(z), \varphi_{-t}(z')) \leqslant \varepsilon \text{ for every } t \geqslant 0\}$$
be the local stable and unstable manifolds at z of size ε, where dist is any smooth distance on M. For $b = s, u$, we define
$$\widetilde{W}_b(z) = TW_b(z,\varepsilon) = \{(z', E_b(z')) \;:\; z' \in W_b(z,\varepsilon)\} \subset G.$$
Finally, for $\omega = (z, E_u(z)) \in \widetilde{K}_u$, set
$$\widetilde{E}_u(\omega) = T_\omega(\widetilde{W}_u(z)),$$
and define $\widetilde{E}_s(\omega)$ as the tangent space at ω of the manifold
$$\widetilde{W}_{s,\mathrm{tot}}(z) = \big\{E \in \pi_G^{-1}(W_s(z,\varepsilon)) : \mathrm{dist}(E_u(z), E) < \varepsilon\big\},$$
where dist is any smooth distance on the fibres of TN.

Lemma 2.1. *For any $\omega = (z, E) \in G$ we have natural isomorphisms*
$$\widetilde{E}_u(\omega) \simeq E_u(z), \quad \widetilde{E}_s(\omega) \simeq E_s(z) \oplus \ker \mathrm{d}\pi_G(\omega).$$
Under these identifications, we have
$$\mathrm{d}\widetilde{\varphi}_t|_{\widetilde{E}_u(\omega)} \simeq \mathrm{d}\varphi_t|_{E_u(z)}, \quad \mathrm{d}\widetilde{\varphi}_t|_{\widetilde{E}_s(\omega)} \simeq \mathrm{d}\varphi_t|_{E_s(z)} \oplus \mathrm{d}\widetilde{\varphi}_t|_{\ker \mathrm{d}\pi_G(\omega)}.$$

Proof. Note that if $\omega = (z, E) \in G$, by (2.6) one has
$$\mathrm{d}\pi_G(\omega) \circ \mathrm{d}\widetilde{\varphi}_t(\omega) = \mathrm{d}(\pi_G \circ \widetilde{\varphi}_t)(\omega) = \mathrm{d}(\varphi_t \circ \pi_G)(\omega) = \mathrm{d}\varphi_t(z) \circ \mathrm{d}\pi_G(\omega). \tag{2.7}$$
This equality shows that $\mathrm{d}\widetilde{\varphi}_t$ preserves $\ker \mathrm{d}\pi_G$. Looking at the definitions of $\widetilde{W}_u(z)$ and $W_u(z,\varepsilon)$, we see that
$$\mathrm{d}\pi_G(\omega)|_{\widetilde{E}_u(z)} : \widetilde{E}_u(z) \to E_u(z)$$
realises an isomorphism. Then by (2.7), it is clear that $\mathrm{d}\pi_G(\omega)|_{T_\omega \widetilde{W}_u(z)}$ realises a conjugation between $\mathrm{d}\widetilde{\varphi}_t(\omega)|_{\widetilde{E}_u(\omega)}$ and $\mathrm{d}\varphi_t(z)|_{E_u(z)}$. Similarly, $\mathrm{d}\pi_G|_{T_\omega \widetilde{W}_s(\omega)}$ realises an isomorphism $T_\omega \widetilde{W}_s(\omega) \simeq E_s(z)$, which conjugates $\mathrm{d}\widetilde{\varphi}_t|_{\widetilde{E}_s(\omega)}$ and $\mathrm{d}\varphi_t|_{E_s(z)}$. Thus the lemma will be proven if we show that we have the direct sum
$$\widetilde{E}_s(z) = T_\omega \widetilde{W}_{s,\mathrm{tot}}(z) = T_\omega \widetilde{W}_s(z) \oplus \ker \mathrm{d}\pi_G(\omega).$$

To see this, take a local trivialization $\widetilde{W}_{s,\mathrm{tot}}(z) \to W_s(z,\varepsilon) \times G_{d-1}(\mathbb{R}^{2d-1})$ sending $\omega \in G$ on (z, E_0) for some $E_0 \in G_{d-1}(\mathbb{R}^{2d-1})$ and such that $\widetilde{W}_s(z)$ is sent to $W_s(z,\varepsilon) \times \{E_0\}$. In these local coordinates one has the identifications

$$T_\omega \widetilde{W}_s(z) \simeq E_s(z) \oplus \{0\} \quad \text{and} \quad \ker \mathrm{d}\pi_G(\omega) \simeq \{0\} \oplus T_{E_0} G_{d-1}(\mathbb{R}^{2d-1}).$$

As $T_\omega \widetilde{W}_{s,\mathrm{tot}}(z)$ is identified with $E_s(z) \oplus T_{E_0} G_{d-1}(\mathbb{R}^{2d-1})$, the proof is complete. □

We conclude this paragraph by noting that for any $\omega = (z, E) \in \widetilde{K}_u$ we have

$$\begin{aligned}
\dim \widetilde{E}_u(\omega) + \dim \widetilde{E}_s(\omega) &= \dim E_u(z) + \dim E_s(z) + \dim \ker \mathrm{d}\pi_G(\omega) \\
&= \dim M - 1 + \dim \pi_G^{-1}(z) \\
&= \dim G - 1,
\end{aligned}$$

since $\dim G = \dim M + \dim \pi_G^{-1}(z)$.

2.6. **Vector bundles.** We define the tautological vector bundle $\mathcal{E} \to G$ by

$$\mathcal{E} = \{(\omega, u) \in \pi_G^*(TN) \ : \ \omega \in G, \ u \in [\omega]\},$$

where $[\omega] = E$ denotes the $(d-1)$ dimensional subspace of $T_{\pi_G(\omega)}N$ represented by $\omega = (z, E)$ and $\pi_G^*(TN)$ is the pullback bundle of TN. Also, we define the "vertical bundle" $\mathcal{F} \to G$ by

$$\mathcal{F} = \{(\omega, W) \in TG \ : \ \mathrm{d}\pi_G(\omega) \cdot W = 0\}.$$

It is a subbundle of the bundle $TG \to G$. The dimensions of the fibres $\mathcal{E}_\omega$ and $\mathcal{F}_\omega$ of $\mathcal{E}$ and $\mathcal{F}$ over ω are given by

$$\dim \mathcal{E}_\omega = d - 1, \quad \dim \mathcal{F}_\omega = \dim \ker \mathrm{d}\pi_G(\omega) = \dim \pi_G^{-1}(z) = d^2 - d$$

for any $\omega \in G$ with $\pi_G(\omega) = z$. Finally, set

$$\mathcal{E}_{k,\ell} = \wedge^k \mathcal{E}^* \otimes \wedge^\ell \mathcal{F}, \quad 0 \leqslant k \leqslant d-1, \quad 0 \leqslant \ell \leqslant d^2 - d,$$

where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$, that is, we replace the fibre $\mathcal{E}_\omega$ by its dual space $\mathcal{E}_\omega^*$. We consider $\mathcal{E}^*$ and not $\mathcal{E}$ since the map $\mathrm{d}\varphi_t(\pi_G(\omega)) : \mathcal{E}_\omega \to \mathcal{E}_{\widetilde{\varphi}_t(\omega)}$ is expanding for $\omega \in \widetilde{K}_u$ and $t \to +\infty$, whereas $\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top} : \mathcal{E}_\omega^* \to \mathcal{E}^*_{\widetilde{\varphi}_t(\omega)}$ is contracting. Here ${}^{-\top}$ denotes the inverse transpose. Indeed, for $\omega = (z, E_u(z)) \in \widetilde{K}_u$ and $u \in E_u(z)^*$ (here $E_u(z)^*$ is the dual vector space of $E_u(z)$) one has

$$\langle \mathrm{d}\varphi_t(z)^{-\top} u, v \rangle = \langle u, \mathrm{d}\varphi_{-t}(\varphi_t(z)) v \rangle, \quad v \in \mathrm{d}\varphi_t(z) E_u(z) = E_u(\varphi_t(z)) \in \mathcal{E}_{\widetilde{\varphi}(t)(\omega)},$$

$\langle ., . \rangle$ being the paring on $\mathcal{E}^*_{\widetilde{\varphi}_t(\omega)}$ and $\mathcal{E}_{\widetilde{\varphi}(t)(\omega)}$. Consequently, the map $\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}$ is contracting on $\mathcal{E}_\omega^*$ when $\omega \in \widetilde{K}_u$, since $\mathrm{d}\varphi_{-t}(\varphi_t(z))$ is contracting on $E_u(\varphi_t(z))$. This fact will be convenient later for the proof of Lemma 3.1 below.

In what follows we use the notation $\omega = (z, \eta) \in G$ and $u \otimes v \in \mathcal{E}_{k,\ell}|_\omega$. By using the flow $\widetilde{\varphi}_t$, we introduce a flow $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ by setting

$$\Phi_t^{k,\ell}(\omega, u \otimes v) = \Big(\widetilde{\varphi}_t(\omega), \ b_t(\omega) \cdot \Big[\big(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\big)^{\wedge k}(u) \otimes \mathrm{d}\widetilde{\varphi}_t(\omega)^{\wedge \ell}(v)\Big]\Big), \tag{2.8}$$

where we set

$$b_t(\omega) = |\det \mathrm{d}\varphi_t(\pi_G(\omega))|_{[\omega]}|^{1/2} \cdot |\det (\mathrm{d}\widetilde{\varphi}_t(\omega)|_{\ker \mathrm{d}\pi_G})|^{-1}.$$

Here the determinants are taken with respect to any choice of smooth metrics g_N on N and the induced metrics g_G on G, as follows. If $\omega = (z, E) \in G$ and $t \in \mathbb{R}$, then the number $|\det \mathrm{d}\varphi_t(z)|_{[\omega]}|$ is defined as the absolute value of the ratio

$$\frac{(\mathrm{d}\varphi_t(z)|_{[\omega]})^{\wedge^{d-1}} \cdot \mu_{[\omega]}}{\mu_{[\widetilde{\varphi}_t(\omega)]}},$$

where $\mu_{[\omega]} = e_{1,[\omega]} \wedge \cdots \wedge e_{d-1,[\omega]} \in \wedge^{d-1}[\omega]$ (resp. $\mu_{[\widetilde{\varphi}_t(\omega))]} \in \wedge^{d-1}[\widetilde{\varphi}_t(\omega)]$) is a volume element given by any basis $e_{1,[\omega]}, \dots, e_{d-1,[\omega]}$ of $[\omega]$ (resp. $[\widetilde{\varphi}_t(\omega)]$) which is orthonormal with respect to the scalar product induced by $g_N|_{[\omega]}$ (resp. $g_N|_{[\widetilde{\varphi}_t(\omega)]}$). The number $|\det (\mathrm{d}\widetilde{\varphi}_t(\omega)|_{\ker \mathrm{d}\pi_G})|$ is defined similarly. If we pass from one orthonormal basis to another one, we multiply the terms by the determinant of a unitary matrix and the absolute value of the above ratio is the same. On the other hand, for a periodic point $\omega_{\widetilde{\gamma}} = \widetilde{\varphi}_{\tau(\gamma)}(\omega_{\widetilde{\gamma}})$ this number is simply $|\det \mathrm{d}\varphi_{\tau(\gamma)}(\pi_G(\omega_{\widetilde{\gamma}}))|_{[\omega_{\widetilde{\gamma}}]}|$. Taking local trivializations of $\mathcal{E}^*$ and $\mathcal{F}$, we see that the action of $\Phi_t^{k,\ell}$ is smooth. Thus we have the following diagram:

$$\begin{array}{ccc}
\mathcal{E}_{k,\ell} & \xrightarrow{\Phi_t^{k,\ell}} & \mathcal{E}_{k,\ell} \\
\big\downarrow & & \big\downarrow \\
G & \xrightarrow{\tilde{\varphi}_t} & G \\
\big\downarrow{\scriptstyle \pi_G} & & \big\downarrow{\scriptstyle \pi_G} \\
N & \xrightarrow{\varphi_t} & N
\end{array}$$

Now, consider the transfer operator

$$\Phi_{-t}^{k,\ell,*} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$$

defined by

$$\Phi_{-t}^{k,\ell,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell}\big[\mathbf{u}(\widetilde{\varphi}_{-t}(\omega))\big], \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}). \tag{2.9}$$

Let $\mathbf{P}_{k,\ell} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$ be the generator of $\Phi_{-t}^{k,\ell,*}$, which is defined by

$$\mathbf{P}_{k,\ell}\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_{-t}^{k,\ell,*}\mathbf{u}\Big)\Big|_{t=0}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

Then we have the equality

$$\mathbf{P}_{k,\ell}(f\mathbf{u}) = (\widetilde{X}f)\mathbf{u} + f(\mathbf{P}_{k,\ell}\mathbf{u}), \quad f \in C^\infty(G), \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}). \tag{2.10}$$

Fix any norm on $\mathcal{E}_{k,\ell}$; this fixes a scalar product on $L^2(G, \mathcal{E}_{k,\ell})$. We also consider the transfer operator $\Phi^{k,\ell,*}$ as a strongly continuous semigroup $e^{-t\mathbf{P}_{k,\ell}}$, $t \geqslant 0$ with generator $\mathbf{P}_{k,\ell}$ with domain in $L^2(G, \mathcal{E}_{k,\ell})$. The exponential ground of the derivatives of φ_{-t} implies an estimate

$$\|\mathrm{e}^{-t\mathbf{P}_{k,\ell}}\|_{L^2(G,\mathcal{E}_{k,\ell}) \to L^2(G,\mathcal{E}_{k,\ell})} \leqslant C\mathrm{e}^{\beta t}, \quad t \geqslant C_0 > 0,$$

for some constants $\beta > 0, C_0 > 0$. Next, we want to study the spectral properties of the operator $\mathbf{P}_{k,\ell}$ applying the work of Dyatlov–Guillarmou [DG16]. For this purpose, one needs to find a neighborhood $\widetilde{V}_u$ of $\widetilde{K}_u$ such that the boundary $\partial \widetilde{V}_u$ has convexity properties with respect to $\widetilde{X}$ (see the condition (2.11) below with $\widetilde{Y}$ replaced by $\widetilde{X}$). However, it is not clear that a such neighborhood exists, and one needs to modify slightly $\widetilde{X}$ outside a neighborhood of $\widetilde{K}_u$ to obtain the desired properties. This is done in §2.7 below.

2.7. **Isolating blocks.** By [CE71, Theorem 1.5], there exists an arbitrarily small open neighborhood $\widetilde{V}_u$ of $\widetilde{K}_u$ in G such that the following holds.

(i) The boundary $\partial \widetilde{V}_u$ of $\widetilde{V}_u$ is smooth,
(ii) The set $\partial_0 \widetilde{V}_u = \{z \in \partial \widetilde{V}_u : \widetilde{X}(z) \in T_z(\partial \widetilde{V}_u)\}$ is a smooth submanifold of codimension 1 of $\partial \widetilde{V}_u$,
(iii) There is $\varepsilon > 0$ such that for any $z \in \partial \widetilde{V}_u$ one has
$$\widetilde{X}(z) \in T_z(\partial \widetilde{V}_u) \quad \Longrightarrow \quad \widetilde{\varphi}_t(z) \notin \operatorname{clos} \widetilde{V}_u, \quad t \in]-\varepsilon, \varepsilon[\setminus \{0\},$$
where clos A denotes the closure of a set A.

In what follows we denote
$$\Gamma_\pm(\widetilde{X}) = \{z \in \widetilde{V}_u : \widetilde{\varphi}_t(z) \in \widetilde{V}_u, \ \mp t > 0\}.$$
A function $\tilde{\rho} \in C^\infty(\operatorname{clos} \widetilde{V}_u, \mathbb{R}_{\geqslant 0})$ will be called a boundary defining function for $\widetilde{V}_u$ if we have $\partial \widetilde{V} = \{z \in \operatorname{clos} \widetilde{V}_u : \tilde{\rho}(z) = 0\}$ and $\mathrm{d}\tilde{\rho}(z) \neq 0$ for any $z \in \partial \widetilde{V}_u$.

By [GMT21, Lemma 2.3] (see also [KSW21, Lemma 5.2]), we have the following result.

Lemma 2.2. *For any small neighborhood $\widetilde{W}_0$ of $\partial_0 \widetilde{V}_u$ in* clos $\widetilde{V}_u$*, we may find a vector field $\widetilde{Y}$ on* clos $\widetilde{V}_u$ *which is arbitrarily close to $\widetilde{X}$ in the C^∞-topology, such that the following holds.*

(1) $\operatorname{supp}(\widetilde{Y} - \widetilde{X}) \subset \widetilde{W}_0$,
(2) $\Gamma_\pm(\widetilde{X}) = \Gamma_\pm(\widetilde{Y})$, *where $\Gamma_\pm(\widetilde{Y})$ is defined as $\Gamma_\pm(\widetilde{X})$ by replacing the flow $(\widetilde{\varphi}_t)$ by the flow generated by $\widetilde{Y}$,*
(3) *For any defining function $\tilde{\rho}$ of $\widetilde{V}_u$ and any $\omega \in \partial \widetilde{V}_u$ we have*
$$\widetilde{Y}\tilde{\rho}(\omega) = 0 \quad \Longrightarrow \quad \widetilde{Y}^2 \tilde{\rho}(\omega) < 0. \tag{2.11}$$

From now on, we will fix $\widetilde{V}_u, \widetilde{W}_0$ and $\widetilde{Y}$ as above. By [DG16, Lemma 1.1] we may find a smooth extension of $\widetilde{Y}$ on G (still denoted by $\widetilde{Y}$) so that for every $\omega \in G$ and $t \geqslant 0$, we have
$$\omega, \widetilde{\varphi}_t(\omega) \in \operatorname{clos} \widetilde{V}_u \quad \Longrightarrow \quad \widetilde{\varphi}_\tau(\omega) \in \operatorname{clos} \widetilde{V}_u \text{ for every } \tau \in [0, t]. \tag{2.12}$$
Let $(\psi_t)_{t \in \mathbb{R}}$ be the flow generated by $\widetilde{Y}$ and let $(\widetilde{\psi}_t)$ be the corresponding flow on G. Set $\widetilde{\Gamma}_\pm = \Gamma_\pm(\widetilde{Y})$ for simplicity. The extended unstable/stable bundles $\widetilde{E}_\pm^* \subset T^* \widetilde{V}_u$ over $\widetilde{\Gamma}_\pm$ are defined by
$$\widetilde{E}_\pm^*(\omega) = \{\Omega \in T_\omega^* \widetilde{V}_u \ : \ \Psi_t(\Omega) \to_{t \to \pm\infty} 0\},$$

where Ψ_t is the symplectic lift of $\widetilde{\psi}_t$, that is

$$\Psi_t(\omega,\Omega) = \Big(\widetilde{\psi}_t(\omega), \mathrm{d}\widetilde{\psi}_t(\omega)^{-\top}\cdot\Omega\Big), \quad (\omega,\Omega)\in T^*G, \quad t\in\mathbb{R},$$

and ${}^{-\top}$ denotes the inverse transpose. Then by [DG16, Lemma 1.10], the bundles $\widetilde{E}_\pm^*(\omega)$ depend continuously on $\omega\in\widetilde{\Gamma}_\pm$, and for any smooth norm $|\cdot|$ on T^*G with some constants $C>0, \beta>0$ independent of ω,Ω for $t\to\mp\infty$ we have

$$|\Psi_{\pm t}(\Omega)| \leqslant C\mathrm{e}^{-\beta|t|}|\Omega|, \quad \Omega\in E_\pm^*(\omega).$$

2.8. **Dyatlov–Guillarmou theory.** Let $\nabla^{k,\ell}$ be any smooth connexion on $\mathcal{E}_{k,\ell}$. Then by (2.10) we have

$$\mathbf{P}_{k,\ell} = \nabla^{k,\ell}_{\widetilde{X}} + \mathbf{A}_{k,\ell}$$

for some $\mathbf{A}_{k,\ell}\in C^\infty(G,\mathrm{End}(\mathcal{E}_{k,\ell}))$. We define a new operator $\mathbf{Q}_{k,\ell}$ by setting

$$\mathbf{Q}_{k,\ell} = \nabla^{k,\ell}_{\widetilde{Y}} + \mathbf{A}_{k,\ell} : C^\infty(G,\mathcal{E}_{k,\ell})\to C^\infty(G,\mathcal{E}_{k,\ell}).$$

Note that $\mathbf{Q}_{k,\ell}$ coincides with $\mathbf{P}_{k,\ell}$ near $\widetilde{K}_u$ since $\widetilde{Y}$ coincides with $\widetilde{X}$ near $\widetilde{K}_u$. Clearly, we have

$$\mathbf{Q}_{k,\ell}(f\mathbf{u}) = (\widetilde{Y}f)\mathbf{u} + f(\mathbf{Q}_{k,\ell}\mathbf{u}), \quad f\in C^\infty(G), \quad \mathbf{u}\in C^\infty(G,\mathcal{E}_{k,\ell}). \tag{2.13}$$

Next, consider the transfer operator $\mathrm{e}^{-t\mathbf{Q}_{k,\ell}} : C^\infty(G,\mathcal{E}_{k,\ell})\to C^\infty(G,\mathcal{E}_{k,\ell})$ with generator $\mathbf{Q}_{k,\ell}$, that is,

$$\partial_t \mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u} = -\mathbf{Q}_{k,\ell}\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u}, \quad \mathbf{u}\in C^\infty(G,\mathcal{E}_{k,\ell}), \quad t\geqslant 0.$$

As above, for some constant $C>0$, we have

$$\|\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\|_{L^2(G,\mathcal{E}_{k,\ell})\to L^2(G,\mathcal{E}_{k,\ell})} \leqslant C\mathrm{e}^{C|t|}, \quad t\geqslant 0.$$

Then for for $\mathrm{Re}(s)>C$, the resolvent $(\mathbf{Q}_{k,\ell}+s)^{-1}$ on $L^2(G,\mathcal{E}_{k,\ell})$ is given by

$$(\mathbf{Q}_{k,\ell}+s)^{-1} = \int_0^\infty \mathrm{e}^{-t(\mathbf{Q}_{k,\ell}+s)}\mathrm{d}t : L^2(G,\mathcal{E}_{k,\ell})\to L^2(G,\mathcal{E}_{k,\ell}). \tag{2.14}$$

Consider the operator

$$\mathbf{R}_{k,\ell}(s) = \mathbf{1}_{\widetilde{V}_u}(\mathbf{Q}_{k,\ell}+s)^{-1}\mathbf{1}_{\widetilde{V}_u}, \quad \mathrm{Re}(s)\gg 1,$$

from $C_c^\infty(\widetilde{V}_u,\mathcal{E}_{k,\ell})$ to $\mathcal{D}'(\widetilde{V}_u,\mathcal{E}_{k,\ell})$, where $\mathcal{D}'(\widetilde{V}_u,\mathcal{E}_{k,\ell})$ denotes the space of $\mathcal{E}_{k,\ell}$-valued distributions. Taking into account (2.11), (2.12) and (2.13), we see that the assumptions (A1)–(A5) in [DG16, §0] are satisfied. We are in position to apply [DG16, Theorem 1] in order to obtain a meromorphic extension of $\mathbf{R}_{k,\ell}(s)$ to the whole plane $\mathbb{C}$. Moreover, according to [DG16, Theorem 2], for every pole $s_0\in\mathbb{C}$ in a small neighborhood of s_0 one has the representation

$$\mathbf{R}_{k,\ell}(s) = \mathbf{R}_{H,k,\ell}(s) + \sum_{j=1}^{J(s_0)} \frac{(-1)^{j-1}(\mathbf{Q}_{k,\ell}+s_0)^{j-1}\Pi_{s_0}^{k,\ell}}{(s-s_0)^j}, \tag{2.15}$$

where $\mathbf{R}_{H,k,\ell}(s) : C_c^\infty(\widetilde{V}_u,\mathcal{E}_{k,\ell})\to\mathcal{D}'(\widetilde{V}_u,\mathcal{E}_{k,\ell})$ is a holomorphic family of operators near $s=s_0$ and $\Pi_{s_0}^{k,\ell} : C_c^\infty(\widetilde{V}_u,\mathcal{E}_{k,\ell})\to\mathcal{D}'(\widetilde{V}_u,\mathcal{E}_{k,\ell})$ is a finite rank projector. Denote by

$K_{\mathbf{R}_{H,k,\ell}(s)}$ and $K_{\Pi^{k,\ell}_{s_0}}$ the Schwartz kernels of the operators $\mathbf{R}_{H,k,\ell}(s)$ and $\Pi^{k,\ell}_{s_0}$, respectively. Recall the definition of the twisted wavefront set

$$\mathrm{WF}'(A) = \{(x, \xi, y, -\eta) : (x, \xi, y, \eta) \in \mathrm{WF}(K_A)\},$$

K_A being the distributional kernel of the operator A. By [DG16, Lemma 3.5], we have

$$\mathrm{WF}'(K_{\mathbf{R}_{H,k,\ell}}(s)) \subset \Delta(T^*\widetilde{V}_u) \cup \Upsilon_+ \cup (\widetilde{E}^*_+ \times \widetilde{E}^*_-). \tag{2.16}$$

Here $\Delta(T^*\widetilde{V}_u)$ is the diagonal in $T^*(\widetilde{V}_u \times \widetilde{V}_u)$,

$$\Upsilon_+ = \{(\Psi_t(\omega, \Omega), \omega, \Omega) \ : \ (\omega, \Omega) \in T^*\widetilde{V}_u, \ t \geqslant 0, \ \langle \widetilde{Y}(\omega), \Omega\rangle = 0\},$$

while the bundles $\widetilde{E}^*_\pm$ and flow Ψ_t are defined in §2.7. Finally, we have

$$\mathrm{supp}(K_{\Pi^{k,\ell}_{s_0}}) \subset \Gamma_+ \times \Gamma_- \quad \text{and} \quad \mathrm{WF}'(K_{\Pi^{k,\ell}_{s_0}}(s)) \subset \widetilde{E}^*_+ \times \widetilde{E}^*_-. \tag{2.17}$$

3. The dynamical zeta function for the Neumann problem

In this section we prove that the function η_{N} admits a meromorphic continuation to the whole complex plane, by relating $\eta_{\mathrm{N}}(s)$ to the flat trace of the cut-off resolvent $\mathbf{R}_{k,\ell}(s)$.

3.1. The flat trace.

First, we recall the definition of the flat trace for operators acting on vector bundles. Consider a manifold V, a vector bundle $\mathcal{E}$ over V and a continuous operator $\mathbf{T} : C^\infty_c(V, \mathcal{E}) \to \mathcal{D}'(V, \mathcal{E})$. Fix a smooth density μ on V; this defines a pairing $\langle\cdot,\cdot\rangle$ on $C^\infty_c(V, \mathcal{E}) \times C^\infty_c(V, \mathcal{E}^*)$. Let

$$K_{\mathbf{T}} \in \mathcal{D}'(V \times V, \mathcal{E} \boxtimes \mathcal{E}^*)$$

be the Schwartz kernel of $\mathbf{T}$ with respect to this pairing, which is defined by

$$\langle K_{\mathbf{T}}, \pi_1^*\mathbf{u} \otimes \pi_2^*\mathbf{v}\rangle = \langle \mathbf{T}\mathbf{u}, \mathbf{v}\rangle, \quad \mathbf{u} \in C^\infty_c(V, \mathcal{E}), \quad \mathbf{v} \in C^\infty_c(V, \mathcal{E}^*),$$

where the pairing on $\mathcal{D}'(V \times V, \mathcal{E} \boxtimes \mathcal{E}^*) \times C^\infty_c(V \times V, \mathcal{E} \boxtimes \mathcal{E}^*)$ is taken with respect to $\mu \times \mu$. Here, the bundle $\mathcal{E} \boxtimes \mathcal{E}^* = \pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^* \to V$ is given by the tensor product of the pullbacks $\pi_1^*\mathcal{E}$, and $\pi_2^*\mathcal{E}^*$, where $\pi_1, \pi_2 : V \times V \to V$ denote the projections over the first and the second factor, respectively.

Denote by $\Delta = \{(x, x) \ : \ x \in V\} \subset V \times V$ the diagonal in $V \times V$ and consider the inclusion map $\iota_\Delta : \Delta \to V \times V, \ (x, x) \mapsto (x, x)$. Assume that

$$\mathrm{WF}'(K_{\mathbf{T}}) \cap \Delta(T^*V \setminus \{0\}) = \emptyset, \tag{3.1}$$

where $\Delta(T^*V \setminus \{0\})$ is the diagonal in $(T^*(V) \setminus \{0\}) \times (T^*(V) \setminus \{0\})$. Then by [Hör90, Theorem 8.2.4], the pull-back

$$\iota_\Delta^* K_{\mathbf{T}} \in \mathcal{D}'(V, \ \mathrm{End}(\mathcal{E}))$$

is well defined, where we used the identification $\iota_\Delta^*(\mathcal{E} \boxtimes \mathcal{E}^*) \simeq \mathcal{E} \otimes \mathcal{E}^* \simeq \mathrm{End}(\mathcal{E})$. If $K_{\mathbf{T}}$ is compactly supported we define the *flat trace* of $\mathbf{T}$ by

$$\mathrm{tr}^\flat\, \mathbf{T} = \langle \mathrm{tr}_{\mathrm{End}(\mathcal{E})}(\iota_\Delta^* K_{\mathbf{T}}), 1\rangle,$$

where again the pairing is taken with respect to μ. It is not hard to see that the flat trace does not depend on the choice of the density μ.

3.2. **The flat trace of cut-off resolvent.** We introduce a cut-off function $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ such that $\widetilde{\chi} \equiv 1$ on $\widetilde{K}_u$. For $\varrho \in C_c^\infty(\mathbb{R}^+ \setminus \{0\})$ define

$$\mathbf{T}_\varrho^{k,\ell}\mathbf{u} := \left(\int_0^\infty \varrho(t)\widetilde{\chi}(\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u})\widetilde{\chi}\mathrm{d}t\right), \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

We may apply the Guillemin trace formula [Gui77, §2 of Lecture 2] (we refer to [BSW21, Lemma 3.1] for a detailed presentation based on the argument of [DZ16, Appendix B]), which implies that the flat trace of $\mathbf{T}_\varrho^{k,\ell}$ is well defined, and

$$\mathrm{tr}^\flat(\mathbf{T}_\varrho^{k,\ell}) = \sum_{\widetilde{\gamma}} \frac{\varrho(\tau_\gamma)\tau^\sharp(\gamma)\,\mathrm{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|}, \tag{3.2}$$

where the sum runs over all periodic orbits $\tilde{\gamma}$ of $(\tilde{\varphi}_t)$. Here,

$$\widetilde{P}_\gamma = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\widetilde{\gamma}})|_{\widetilde{E}_u(\omega_{\widetilde{\gamma}})\oplus\widetilde{E}_s(\omega_{\widetilde{\gamma}})}$$

is the linearized Poincaré map of the closed orbit

$$t \mapsto \tilde{\gamma}(t) = (\gamma(t), E_u(\gamma(t)))$$

of the flow $(\widetilde{\varphi}_t)$ and $\omega_{\tilde{\gamma}} \in \mathrm{Im}(\tilde{\gamma})$ is any reference point taken in the image of $\tilde{\gamma}$. Note that if we take another point $\omega'_{\tilde{\gamma}} \in \mathrm{Im}(\tilde{\gamma})$, then the map $\mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega'_{\widetilde{\gamma}})$ is conjugated to $\mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\widetilde{\gamma}})$ by $\mathrm{d}\widetilde{\varphi}_{t_1}(\omega_{\widetilde{\gamma}})$, where $t_1 \in \mathbb{R}$ is chosen so that $\widetilde{\varphi}_{t_1}(\omega'_{\widetilde{\gamma}}) = \omega_{\widetilde{\gamma}}$. Hence the determinant $\det(\mathrm{Id} - P_\gamma)$ does not depend on the reference point $\omega_{\widetilde{\gamma}}$ and is well defined. The number $\mathrm{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})$ is the trace of the linear map

$$\alpha_{\omega_{\widetilde{\gamma}},\tau(\gamma)}^{k,\ell} : \mathcal{E}_{k,\ell}|_{\omega_{\widetilde{\gamma}}} \to \mathcal{E}_{k,\ell}|_{\omega_{\widetilde{\gamma}}},$$

where for $t \in \mathbb{R}$ and $\omega \in G$, we denote by

$$\alpha_{\omega,t}^{k,\ell} : \mathcal{E}_{k,\ell}|_\omega \to \mathcal{E}_{k,\ell}|_{\widetilde{\varphi}_t(\omega)}$$

the restriction of the map $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ to the fiber $\mathcal{E}_{k,\ell}|_\omega$. Again, if we take another reference point $\omega'_{\widetilde{\gamma}}$, the map $\alpha_{\omega'_{\widetilde{\gamma}},\tau(\gamma)}^{k,\ell}$ is conjugated to $\alpha_{\omega_{\widetilde{\gamma}},\tau(\gamma)}^{k,\ell}$, hence its trace depends only on $\widetilde{\gamma}$, and this justifies the notation $\mathrm{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})$.

Next, we follow the strategy of [BSW21, §4.1] which is based on that used in [DZ16, §4] for Anosov flows on closed manifolds to compute the flat trace of the (shifted) resolvent defined below. We may apply formula (3.2) with the functions $\varrho_{s,T}(t) = \mathrm{e}^{-st}\varrho_T(t)$, where $\varrho_T \in C_c^\infty(\mathbb{R}^+)$ satisfies $\mathrm{supp}\,\varrho_T \subset [\varepsilon/2, T+1]$ for $0 < \varepsilon < d_0 = \min_{\gamma\in\mathcal{P}} \tau(\gamma)$ small and $\varrho_T \equiv 1$ on $[\varepsilon, T]$. Then taking the limit $T \to \infty$, we obtain, with (2.14) in mind,

$$\mathrm{tr}^\flat\,\mathbf{R}_\varepsilon^{k,\ell}(s) = \sum_{\tilde{\gamma}} \frac{\mathrm{e}^{-s\tau(\gamma)}\tau^\sharp(\gamma)\,\mathrm{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|}, \quad \mathrm{Re}(s) \gg 1. \tag{3.3}$$

Here for $\mathrm{Re}(s)$ large enough and $\varepsilon > 0$ small, we set

$$\mathbf{R}_\varepsilon^{k,\ell}(s) := \widetilde{\chi}\mathrm{e}^{-\varepsilon(s+\mathbf{Q}_{k,\ell})}(\mathbf{Q}_{k,\ell} + s)^{-1}\widetilde{\chi},$$

and ε is chosen so that $\mathrm{e}^{-\varepsilon \mathbf{Q}_{k,\ell}} \operatorname{supp}(\widetilde{\chi}) \subset \widetilde{V}_u$, so that $\mathbf{R}_\varepsilon^{k,\ell}(s)$ is well defined. The equality (3.3) is exactly Equation (4.21) in [BSW21], and we refer to the aforecited work for a detailed proof of this identity. Note that the flat trace $\operatorname{tr}^\flat \mathbf{R}_\varepsilon^{k,\ell}(s)$ is well defined thanks to the information of the wavefront set $\mathrm{WF}'(K_{\mathbf{R}_\varepsilon^{k,\ell}(s)})$ obtained from (2.16), together with the multiplication properties satisfied by wavefront sets (see [Hör90, Theorem 8.2.14]).

Next, one states the following result, similar to that in [FT17, Section 2]. This crucial lemma explains the reason to introduce the bundles $\mathcal{E}_{k,\ell}$. For the sake of completeness, we present a detailed proof.

Lemma 3.1. *For any periodic orbit $\tilde{\gamma}$ related to a periodic orbit γ, we have*

$$\frac{1}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d^2-d} (-1)^{k+\ell} \operatorname{tr}(\alpha_{\tilde{\gamma}}^{k,\ell}) = |\det(\mathrm{Id} - P_\gamma)|^{-1/2}.$$

Proof. Let $\gamma(t)$ be a periodic orbit and let $\tilde{\gamma}(t) = (\gamma(t), E_u(\gamma(t)))$, $\omega_{\tilde{\gamma}} \in \tilde{\gamma}$, $z \in \gamma$. Set

$$P_{\gamma,u} = \mathrm{d}\varphi_{-\tau(\gamma)}(z)|_{E_u(z)}, \quad P_{\gamma,s} = \mathrm{d}\varphi_{-\tau(\gamma)}(z)|_{E_s(z)},$$
$$P_{\gamma,\perp} = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})|_{\ker \mathrm{d}\pi_G(\omega)}, \quad P_{\gamma,\perp}^{-1} = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})^{-1}|_{\ker \mathrm{d}\pi_G(\omega)}.$$

The linearized Poincaré map $\widetilde{P}_\gamma$ of the closed orbit $\tilde{\gamma}$ satisfies

$$\begin{aligned} \det(\mathrm{Id} - \widetilde{P}_\gamma) &= \det\left(\mathrm{Id} - \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}|_{\widetilde{E}_s(\omega)\oplus \widetilde{E}_u(\omega)}\right) \\ &= \det\left(\mathrm{Id} - P_\gamma\right) \det\left(\mathrm{Id} - P_{\gamma,\perp}\right) \end{aligned} \tag{3.4}$$

since $\widetilde{E}_s(\omega) \simeq E_s(z) \oplus \ker \mathrm{d}\pi_G(\omega)$ and $\widetilde{E}_u(\omega) \simeq E_u(z)$ by Lemma 2.1. Recall the well known formula

$$\det(\mathrm{Id} - A) = \sum_{j=0}^{k} (-1)^j \operatorname{tr} \wedge^j A$$

for any endomorphism A of a k-dimensional vector space. Moreover, notice that

$$\operatorname{tr}(\alpha_{\tilde{\gamma}}^{k,\ell}) = \operatorname{tr} \wedge^k P_{\gamma,u} \operatorname{tr} \wedge^\ell P_{\gamma,\perp}^{-1},$$

since by (2.8), $\alpha_{\tilde{\gamma}}^{k,\ell}$ coincides with the map

$$\wedge^k \left[\mathrm{d}\varphi_{\tau(\gamma)}(\pi_G(\omega_{\tilde{\gamma}}))^{-\top}\right] \otimes \wedge^\ell \left[\mathrm{d}\varphi_{\tau(\gamma)}(\omega_{\tilde{\gamma}})\right] : \wedge^k \mathcal{E}^*|_{\omega_{\tilde{\gamma}}} \otimes \wedge^\ell \mathcal{F}|_{\omega_{\tilde{\gamma}}} \to \wedge^k \mathcal{E}^*|_{\omega_{\tilde{\gamma}}} \otimes \wedge^\ell \mathcal{F}|_{\omega_{\tilde{\gamma}}}.$$

Therefore, one gets

$$\begin{aligned} &\sum_{\ell=0}^{d^2-d} \sum_{k=0}^{d-1} (-1)^{k+\ell} \operatorname{tr}(\alpha_{\tilde{\gamma}}^{k,\ell}) \\ &\qquad = b_{\tau(\gamma)}(\omega_{\tilde{\gamma}}) \left(\sum_{k=0}^{d-1} (-1)^k \operatorname{tr} \wedge^k P_{\gamma,u}\right) \left(\sum_{\ell=0}^{d^2-d} (-1)^\ell \operatorname{tr} \wedge^\ell P_{\gamma,\perp}^{-1}\right) \\ &\qquad = |\det(P_{\gamma,u})|^{-1/2} |\det(P_{\gamma,\perp})| \det(\mathrm{Id} - P_{\gamma,u}) \det(\mathrm{Id} - P_{\gamma,\perp}^{-1}). \end{aligned} \tag{3.5}$$

Here we have used the equality

$$\begin{aligned} b_{\tau(\gamma)}(\omega_{\tilde\gamma}) &= |\det \mathrm{d}\varphi_{\tau(\gamma)}(\pi_G(\omega_{\tilde\gamma}))|_{[\omega_{\tilde\gamma}]}|^{1/2} \cdot |\det\left(\mathrm{d}\widetilde{\varphi}_{\tau(\gamma)}(\omega_{\tilde\gamma})|_{\ker \mathrm{d}\pi_G}\right)|^{-1} \\ &= |\det(P_{\gamma,u})|^{-1/2} |\det(P_{\gamma,\perp})| \end{aligned}$$

which holds because $P_{\gamma,u}$ and $P_{\gamma,\perp}$ are defined with $\mathrm{d}\varphi_{-t}$ and $\mathrm{d}\tilde\varphi_{-t}$, respectively. Therefore (3.4) yields

$$\sum_{k,\ell}(-1)^{k+\ell}\frac{\mathrm{tr}(\alpha_{\tilde\gamma}^{k,\ell})}{|\det(\mathrm{Id}-\widetilde{P}_\gamma)|} = \frac{\det(\mathrm{Id}-P_{\gamma,u})\det(\mathrm{Id}-P_{\gamma,\perp}^{-1})|\det(P_{\gamma,u})|^{-1/2}}{|\det(\mathrm{Id}-P_\gamma)||\det(\mathrm{Id}-P_{\gamma,\perp})||\det(P_{\gamma,\perp})|^{-1}}. \tag{3.6}$$

Since P_γ is a linear symplectic map, we have

$$\det(\mathrm{Id}-P_{\gamma,s}^{-1}) = \det(\mathrm{Id}-P_{\gamma,u}),\ \det(P_{\gamma,s}) = \det(P_{\gamma,u}^{-1}),$$

and one deduces

$$\begin{aligned} |\det(\mathrm{Id}-P_\gamma)| &= |\det(\mathrm{Id}-P_{\gamma,u})||\det(\mathrm{Id}-P_{\gamma,s})| \\ &= |\det(P_{\gamma,s})||\det(\mathrm{Id}-P_{\gamma,u})||\det(\mathrm{Id}-P_{\gamma,s}^{-1})| \\ &= |\det(P_{\gamma,u})|^{-1}|\det(\mathrm{Id}-P_{\gamma,u})|^2. \end{aligned}$$

For $t>0$ the map $\mathrm{d}\widetilde{\varphi}_t = (\mathrm{d}\widetilde{\varphi}_{-t})^{-1}$ is contracting on $\ker \mathrm{d}\pi_G \subset \tilde{E}_s(\omega_{\tilde\gamma})$ (resp. $\mathrm{d}\varphi_{-t}$ is contracting on $E_u(z)$) and these contractions yield $\det(\mathrm{Id}-P_{\gamma,\perp}^{-1}) > 0$ (resp. $\det(\mathrm{Id}-P_{\gamma,u}) > 0$). Thus the terms involving $P_{\gamma,\perp}$ in (3.6) cancel and since

$$|\det(\mathrm{Id}-P_\gamma)|^{-1/2} = |\det(P_{\gamma,u})|^{1/2}\det(\mathrm{Id}-P_{\gamma,u})^{-1},$$

the right hand side of (3.6) is equal to $|\det(\mathrm{Id}-P_\gamma)|^{-1/2}$. □

3.3. **Meromorphic continuation of η_{N}.** From Lemma 3.1 and (3.3), we deduce that for $\mathrm{Re}(s) \gg 1$, we have

$$\eta_{\mathrm{N}}(s) = \sum_{k=0}^{d-1}\sum_{\ell=0}^{d^2-d}(-1)^{k+\ell}\,\mathrm{tr}^\flat\,\mathbf{R}_\varepsilon^{k,\ell}(s),$$

where $\eta_{\mathrm{N}}(s)$ is defined by

$$\eta_{\mathrm{N}}(s) = \sum_\gamma \frac{\tau^\sharp(\gamma)\mathrm{e}^{-\tau(\gamma)s}}{|\det(1-P_\gamma)|^{1/2}}.$$

Since for every k,ℓ the family $s \mapsto \mathbf{R}_\varepsilon^{k,\ell}(s)$ extends to a meromorphic family on the whole complex plane, so does $s \mapsto \eta_{\mathrm{N}}(s)$. Indeed, it follows from the proof of [DG16, Lemma 3.2] that $s \mapsto K_{\mathbf{R}_\varepsilon^{k,\ell}(s)}$ is continuous as a map[1]

$$\mathbb{C} \setminus \mathrm{Res}(\mathbf{R}_\varepsilon^{k,\ell}) \to \mathcal{D}'_\Gamma(G\times G, \mathcal{E}_{k,\ell}\boxtimes\mathcal{E}_{k,\ell}^*).$$

[1]This follows from the fact that the estimates on the wavefront set of $\mathbf{R}_\varepsilon^{k,\ell}(s)$ given in [DG16, Lemma 3.5] are locally uniform with respect to $s \in \mathbb{C}$.

Here for $s \notin \mathrm{Res}(\mathbf{R}_\varepsilon^{k,\ell})$ the distribution $K_{\mathbf{R}_\varepsilon^{k,\ell}(s)}$ is the Schwartz kernel of $\mathbf{R}_\varepsilon^{k,\ell}(s)$ and (see (2.16) for the notation)

$$\Gamma = \Delta_\varepsilon \cup \Upsilon_{+,\varepsilon} \cup \widetilde{E}_+^* \times \widetilde{E}_-^*,$$

where $\Delta_\varepsilon = \{(\Psi_\varepsilon(\omega,\Omega),\omega,\Omega) : (\omega,\Omega) \in T^*(\widetilde{V}_u) \setminus \{0\}\}$ and

$$\Upsilon_{+,\varepsilon} = \{(\Psi_t(\omega,\Omega),\omega,\Omega) : (\omega,\Omega) \in T^*(\widetilde{V}_u) \setminus \{0\},\ t \geqslant \varepsilon,\ \langle \widetilde{Y}(\omega),\Omega\rangle = 0\},$$

while $\mathcal{D}'_\Gamma(G \times G, \mathcal{E}_{k,\ell} \boxtimes \mathcal{E}_{k,\ell}^*)$ is the space of distributions valued in $\mathcal{E}_{k,\ell} \boxtimes \mathcal{E}_{k,\ell}^*$ whose wavefront set is contained in Γ. This space is endowed with its usual topology (see [Hör90, §8.2]). Thus, outside the set of poles $\mathrm{Res}(\mathbf{R}_\varepsilon^{k,\ell})$, we apply the procedure with a flat trace. In particular, $s \mapsto \mathrm{tr}^\flat\, \mathbf{R}_\varepsilon^{k,\ell}(s)$ is continuous on $\mathbb{C} \setminus \mathrm{Res}(\mathbf{R}_\varepsilon^{k,\ell})$ by [Hör90, Theorem 8.2.4]. Finally, Cauchy's formula implies that this map is meromorphic on $\mathbb{C}$ and this completes the proof that the Dirichlet series $\eta_{\mathrm{N}}(s)$ admits a meromorphic continuation in $\mathbb{C}$.

Next, we establish that $\eta_{\mathrm{N}}(s)$ has simple poles with integer residues. To do this, we may proceed as in [DG16, §4]. For the sake of completeness we reproduce the argument. Let $s_0 \in \mathrm{Res}(\mathbf{R}_{k,\ell})$ for some k,ℓ. Recalling the development (2.15), it is enough to show that

$$\mathrm{tr}^\flat \left(\widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q}_{k,\ell})} \left[(\mathbf{Q}_{k,\ell}+s_0)^{j-1}\Pi_{s_0}^{k,\ell}\right] \widetilde{\chi}\right) = 0, \quad j \geqslant 2, \tag{3.7}$$

and

$$\mathrm{tr}^\flat \left(\widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q}_{k,\ell})} \Pi_{s_0}^{k,\ell} \widetilde{\chi}\right) = \mathrm{rank}\ \Pi_{s_0}^{k,\ell}. \tag{3.8}$$

In the following we fix k and ℓ. We may write

$$\Pi_{s_0}^{k,\ell} = \sum_{i=1}^m \mathbf{u}_i \otimes \mathbf{v}_i,$$

where $\otimes$ denotes the Hilbert tensor product and by (2.17) for $i = 1,\dots,m$ we have

$$\begin{aligned} &\mathbf{u}_i \in \mathcal{D}'(\tilde{V}_u, \mathcal{E}_{k,\ell}), \quad \mathrm{supp}(\mathbf{u}_i) \subset \Gamma_+, \quad \mathrm{WF}'(\mathbf{u}_i) \subset \tilde{E}_+^*, \\ &\mathbf{v}_i \in \mathcal{D}'(\tilde{V}_u, \mathcal{E}_{k,\ell}^*), \quad \mathrm{supp}(\mathbf{v}_i) \subset \Gamma_-, \quad \mathrm{WF}'(\mathbf{v}_i) \subset \tilde{E}_-^*. \end{aligned} \tag{3.9}$$

The relations

$$\widetilde{E}_+^* \cap \widetilde{E}_-^* \cap (T^*(\widetilde{V}) \setminus \{0\}) = \emptyset,$$

make possible to define the pairing $\langle \mathbf{u}_i, \mathbf{v}_p\rangle$ on $\mathcal{E}_{k,\ell} \times \mathcal{E}_{k,\ell}^*$ for $i,p = 1,\dots,m$ which yields a distribution on $\tilde{V}_u$. This distribution is compactly supported since

$$\mathrm{supp}\, \mathbf{u}_i \cap \mathrm{supp}\, \mathbf{v}_p \subset \Gamma_+ \cap \Gamma_- = \widetilde{K}_u.$$

The family $(\mathbf{u}_i)$ is a basis of the range of $\Pi_{s_0}^{k,\ell}$. By definition of the flat trace and the information on the wavefront sets, we can write

$$\begin{aligned} &\mathrm{tr}^\flat \big(\widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q}_{k,\ell})} \left[(\mathbf{Q}_{k,\ell}+s_0)^{j-1}\Pi_{s_0}^{k,\ell}\right] \widetilde{\chi}\big) \\ &\qquad = \sum_{i=1}^m \int_{\widetilde{V}_u} \langle \widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q})} (\mathbf{Q}_{k,\ell}+s_0)^{j-1} \mathbf{u}_i, \widetilde{\chi} \mathbf{v}_i \rangle. \end{aligned} \tag{3.10}$$

Here the integrals make sense taking into account the estimates of the supports and the wavefront sets of $\mathbf{u}_i$ and $\mathbf{v}_p$ mentioned above. Since $\Pi_{s_0}^{k,\ell} \circ \Pi_{s_0}^{k,\ell} = \Pi_{s_0}^{k,\ell}$, the family $(\mathbf{v}_p)$ is dual to the basis $(\mathbf{u}_i)$ in the sense that

$$\int_{\widetilde{V}_u} \langle \mathbf{u}_i, \mathbf{v}_p \rangle = \delta_{ip}, \quad 1 \leqslant i, p \leqslant m, \tag{3.11}$$

where δ_{ip} are the Kronecker symbols. Introduce

$$C_{s_0,k,\ell}^{(j)} = \left\{ \mathbf{u} \in \mathcal{D}'(\tilde{V}_u, \mathcal{E}_{k,\ell}) : \operatorname{supp} u \subset \Gamma_+, \ \mathrm{WF}(u) \subset \tilde{E}_+^*, \ (\mathbf{Q}_{k,\ell} + s_0)^j \mathbf{u} = 0 \right\}.$$

Then, since $\widetilde{\chi} = 1$ near $\widetilde{K}_u$, one deduces that $\widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0 + \mathbf{Q}_{k,\ell})} (\mathbf{Q}_{k,\ell} + s_0) \Pi_{s_0}^{k,\ell} \widetilde{\chi}$ maps

$$\widetilde{\chi} C_{s_0,k,\ell}^{(j+1)} \to \widetilde{\chi} C_{s_0,k,\ell}^{(j)}, \quad \text{and} \quad \widetilde{\chi} C_{s_0,k,\ell}^{(1)} \to \{0\}, \quad j \geqslant 1.$$

This fact and (3.11) show that (3.7) holds. To prove (3.8), we write

$$\sum_i \int_{\widetilde{V}_u} \langle \widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0 + \mathbf{Q}_{k,\ell})} \mathbf{u}_i, \widetilde{\chi} \mathbf{v}_i \rangle = \sum_i \int_{\widetilde{V}_u} \langle \mathbf{u}_i, \mathbf{v}_i \rangle - \sum_i \int_0^\varepsilon \mathrm{d}t \int_{\widetilde{V}_u} \left\langle \widetilde{\chi} \mathrm{e}^{-t(s_0 + \mathbf{Q}_{k,\ell})} (\mathbf{Q}_{k,\ell} + s_0) \mathbf{u}_i, \widetilde{\chi} \mathbf{v}_i \right\rangle.$$

Now, we replace ε by t in (3.7) for any $t \in [0, \varepsilon]$, and we obtain that the last sum in the right hand side of the above equation vanishes. Finally, applying (3.11), we obtain (3.8).

4. The dynamical zeta function for particular rays

In this section we adapt the above construction to prove the following result.

Theorem 4. *Let $q \in \mathbb{N}_{\geqslant 1}$. The function $\eta_q(s)$ defined by*

$$\eta_q(s) = \sum_{\gamma \in \mathcal{P}, \, m(\gamma) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma) \mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \quad \mathrm{Re}(s) \gg 1,$$

where the sum runs over all periodic rays γ with $m(\gamma) \in q\mathbb{N}$, admits a meromorphic continuation to the whole complex plane with simple poles and residues valued in $\mathbb{Z}/q$.

Note that for large $\mathrm{Re}(s)$ we have the formula

$$\eta_{\mathrm{D}}(s) = 2\eta_2(s) - \eta_{\mathrm{N}}(s). \tag{4.1}$$

In particular, Theorem 4 implies that $\eta_{\mathrm{D}}(s)$ also extends meromorphically to the whole complex plane, since $\eta_{\mathrm{N}}(s)$ does by the preceding section. In particular, we obtain Theorem 1 since $2\eta_2(s)$ has simple poles with residues in $\mathbb{Z}$.

4.1. **The q-reflection bundle.** For $q \geqslant 2$ define the *q-reflection bundle* $\mathcal{R}_q \to M$ by

$$\mathcal{R}_q = \Big(\Big[S\mathbb{R}^d \setminus \Big(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\Big)\Big] \times \mathbb{R}^q\Big)\Big/ \approx, \tag{4.2}$$

where the equivalence classes of the relation $\approx$ are defined as follows. For $(x, v) \in S\mathbb{R}^d \setminus \Big(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\Big)$ and $\xi \in \mathbb{R}^q$, we set

$$[(x, v, \xi)] = \{(x, v, \xi), (x, v', A(q) \cdot \xi)\} \quad \text{ if } (x, v) \in \mathcal{D}_{\mathrm{in}},\ (x, v') \in \mathcal{D}_{\mathrm{out}},$$

where $A(q)$ is the $q \times q$ matrix with entries in $\{0, 1\}$ given by

$$A(q) = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Clearly, the matrix $A(q)$ yields a shift permutation

$$A(q)(\xi_1, \xi_2, ..., \xi_q) = (\xi_q, \xi_1, ..., \xi_{q-1}).$$

This indeed defines an equivalence relation since $(x, v') \in \mathcal{D}_{\mathrm{out}}$ whenever $(x, v) \in \mathcal{D}_{\mathrm{in}}$. Note that

$$A(q)^q = \mathrm{Id}, \quad \operatorname{tr} A(q)^j = 0, \quad j = 1, \dots, q-1. \tag{4.3}$$

Let us describe the smooth structure of $\mathcal{R}_q$, using the charts of M and the notations of §2.2. For $z_\star \in \mathcal{D}_{\mathrm{in}}$, let $U_{z_\star} = B(0, \delta) = \{x \in \mathbb{R}^{d-1} : |x| < \delta\}$ be a neighborhood of 0 used for the definition of $F_{z_\star}$ (see §2.2) and let

$$\Psi_{z_\star}^{-1} : \mathcal{O}_{z_\star} \to]-\epsilon, \epsilon[\times B(0, \delta) \times B(0, \delta) = W_{z_\star}$$

be a chart. Then the bundle $\mathcal{R}_q \to M$ can be defined by defining its transition maps, as follows. Let $W = \Psi^{-1}(B \setminus \pi^{-1}(\partial D))$ be a chart. In the smooth coordinates introduced in §2.2, we have $W_{z_\star} \cap W = W_+ \sqcup W_-$, where

$$W_+ =]0, \varepsilon[\times B^{2d-2}(0, \delta) \quad \text{ and } \quad W_- =]-\varepsilon, 0[\times B^{2d-2}(0, \delta).$$

Then we define the transition map $\alpha_{z_\star} : W_{z_\star} \cap W \to \mathrm{GL}(\mathbb{R}^q)$ of the bundle $\mathcal{R}_q$ with respect to the pair of charts $(\Psi_{z_\star}, \Psi)$ to be the locally constant map defined by

$$\alpha_{z_\star}(z) = \begin{cases} \mathrm{Id} & \text{if } z \in W_-, \\ A(q) & \text{if } z \in W_+. \end{cases}$$

For $z_\star, z'_\star \in \mathcal{D}_{\mathrm{in}}$, the transition map of $\mathcal{R}_q$ for the pair of charts $(\Psi_{z_\star}, \Psi_{z'_\star})$ is declared to be constant and equal to Id on $W_{z_\star} \cap W_{z'_\star}$. In this way we obtain a smooth bundle $\mathcal{R}_q$ over M, which is clearly homeomorphic to the quotient space (4.2). Since the transition maps of $\mathcal{R}_q$ are locally constant, there is a natural flat connexion d^q on $\mathcal{R}_q$ which is given in the charts by the trivial connexion on $\mathbb{R}^q$.

Consider a small smooth neighborhood V of K. As in §2.4, we embed V into a smooth compact manifold without boundary N, and we fix an extension of $\mathcal{R}_q$ to N

(this is always possible if we choose N to be the double manifold of V). Consider any connexion ∇^q on the extension of $\mathcal{R}_q$ which coincides with d^q near K, and denote by

$$P_{q,t}(z) : \mathcal{R}_q(z) \to \mathcal{R}_q(\varphi_t(z))$$

the parallel transport of ∇^q along the curve $\{\varphi_\tau(z) \ : \ 0 \leqslant \tau \leqslant t\}$. We have a smooth action of φ_t^q on $\mathcal{R}_q$ which is given by the horizontal lift of φ_t

$$\varphi_t^q(z,\xi) = (\varphi_t(z), P_{q,t}(z)\cdot\xi), \quad (z,\xi) \in \mathcal{R}_q.$$

As in (3.2) we see that for a periodic orbit γ we define $P_{q,\gamma}$ as an endomorphism on $\mathbb{R}^q$. From (4.3), and the fact that ∇^q coincides with d^q near K, we easily deduce that for any periodic orbit $\gamma = (\varphi_\tau(z))_{\tau\in[0,\tau(\gamma)]}$, we have

$$\operatorname{tr} P_{q,\gamma} = \begin{cases} q & \text{if} \quad m(\gamma) = 0 \mod q, \\ 0 & \text{if} \quad m(\gamma) \neq 0 \mod q. \end{cases} \tag{4.4}$$

4.2. **Transfer operators acting on G.** Now, consider the bundle

$$\mathcal{E}_{k,\ell}^q = \mathcal{E}_{k,\ell} \otimes \pi_G^* \mathcal{R}_q,$$

where $\pi_G^*\mathcal{R}_q$ is the pullback of $\mathcal{R}_q$ by π_G and $\mathcal{E}_{k,\ell}$ is defined in §2.6, so that $\pi_G^*\mathcal{R}_q \to G$ is a vector bundle over G. We may lift the flow φ_t^q to a flow $\Phi_t^{k,\ell,q}$ on $\mathcal{E}_{k,\ell}^q$ which is defined locally near $\widetilde{K}_u$ by

$$\begin{aligned}\Phi_t^{k,\ell,q}&(\omega, u\otimes v\otimes\xi) \\ &= \left(\widetilde{\varphi}_t(\omega),\ b_t(\omega)\cdot\left[\left(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\right)^{\wedge k}(u)\otimes(\mathrm{d}\widetilde{\varphi}_t(\omega))^{\wedge\ell}(v)\otimes P_{q,t}(z)\cdot\xi\right]\right)\end{aligned}$$

for any $\omega = (z,E) \in G$, $u\otimes v\otimes\xi \in \mathcal{E}_{k,\ell}^q(\omega)$ and $t \in \mathbb{R}$. Here $b_t(\omega)$ is defined in §2.6. As in §2.8, we consider a smooth connexion $\nabla^{k,\ell,q} = \nabla^{k,\ell}\otimes\pi_G^*\nabla^q$ on $\mathcal{E}_{k,\ell}^q$. Define the transfer operator

$$\Phi_{-t}^{k,\ell,q,*} : C^\infty(G, \mathcal{E}_{k,l}^q) \to C^\infty(G, \mathcal{E}_{k,\ell}^q)$$

by

$$\Phi_{-t}^{k,\ell,q,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell,q}[\mathbf{u}(\tilde{\varphi}_{-t}(\omega)],\ \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}^q).$$

Then the operator

$$\mathbf{P}_{k,\ell,q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{-t}^{k,\ell,q,*}\right)\Big|_{t=0},\ \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}^q).$$

which is defined near $\widetilde{K}_u$, can be written locally as $\nabla_{\widetilde{X}}^{k,\ell,q} + \mathbf{A}_{k,\ell,q}$ for some $\mathbf{A}_{k,\ell,q} \in C^\infty(\widetilde{U}_u, \operatorname{End}\mathcal{E}_{k,\ell}^q)$ which is defined in some small neighborhood $\widetilde{U}_u$ of $\widetilde{K}_u$. Next, we choose some $\mathbf{B}_{k,\ell,q} \in C^\infty(G, \operatorname{End}\mathcal{E}_{k,\ell}^q)$ which coincides $\mathbf{A}_{k,\ell,q}$ near $\widetilde{K}_u$. We consider $\widetilde{V}_u$ and $\widetilde{Y}$ as in §2.7, and set

$$\mathbf{Q}_{k,\ell,q} = \nabla_{\widetilde{Y}}^{k,\ell,q} + \mathbf{B}_{k,\ell,q} : C^\infty(G, \mathcal{E}_{k,\ell}^q) \to C^\infty(G, \mathcal{E}_{k,\ell}^q).$$

4.3. **Meromorphic continuation of** $\eta_q(s)$**.** For $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ such that $\widetilde{\chi} \equiv 1$ near $\widetilde{K}_u$, we define

$$\mathbf{R}_\varepsilon^{k,\ell,q}(s) := \widetilde{\chi} \mathrm{e}^{-\varepsilon(\mathbf{Q}_{k,\ell,q}+s)}(\mathbf{Q}_{k,\ell,q}+s)^{-1}\widetilde{\chi}$$

Repeating the argument of the preceding section, one can obtain an analog of (3.3) where the factor $\operatorname{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})$ must be replaced by $\operatorname{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})\operatorname{tr}(P_{q,\gamma})$. This leads a meromorphic continuation of $\mathbf{R}_\varepsilon^{k,\ell,q}(s)$.

On the other hand, by (4.4) one gets $\operatorname{tr}(P_{q,\gamma}) = \mathbf{1}_{q\mathbb{N}}(m(\gamma))$. In particular, proceeding exactly as in the the preceding section, we obtain that for $\operatorname{Re}(s)$ large enough,

$$\sum_{k,\ell}(-1)^{k+\ell}\operatorname{tr}^\flat \mathbf{R}_\varepsilon^{k,\ell,q}(s) = q \sum_{\substack{\gamma\in\mathcal{P} \\ m(\gamma)\in q\mathbb{N}}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}}. \tag{4.5}$$

Therefore, repeating the argument of §3, we establish a meromorphic continuation of the function $s \mapsto \eta_q(s)$. Finally, by using (4.5), we may proceed exactly as in §3.3 to show that $q\eta_q$ has integer residues. This completes the proof of Theorem 4.

5. The modified Lax–Phillips conjecture for real analytic obstacles

In this section, we assume that the obstacles $D_1, \dots, D_r$ have real analytic boundary. Then the smooth structure on M defined in §2.2 induces an analytic structure on M. Indeed, with notations of §2.2, the local parametrizations $F_{z_\star}$ of $\mathcal{D}_{\mathrm{in}}$ can be chosen to be real analytic, as $\mathcal{D}_{\mathrm{in}}$ is a real analytic submanifold of $S\mathbb{R}^{d-1}$. This makes the transition maps (2.3) real analytic, and thus we obtain a real analytic structure on M. In the charts defined by $\Psi_{z_\star}$ and Ψ (see §4.1), the billiard flow φ_t is a translation and it defines a real analytic flow. Of course, the Grassmannian bundle $G \to M$ also becomes real analytic. Consequently, the lifted flow $\widetilde{\varphi}_t$ on G, which is defined by (2.6), is real analytic as well.

Consider the bundles $\mathcal{E}_{k,\ell}^q \to G$ defined in §4.2 for $q \geqslant 2$, $1 \leqslant k \leqslant d-1$ and $1 \leqslant \ell \leqslant d^2-d$. In the case $q=1$ the bundles $\mathcal{E}_{k,\ell}^1 \to G$ are isomorphic to $\mathcal{E}_{k,\ell}$, $\mathcal{E}_{k,\ell}$ being the bundles defined in §3. As before, we naturally extend the flow $\widetilde{\varphi}_t$ to a flow $\Phi_t^{k,\ell,q}$ (which is non complete) on $\mathcal{E}_{k,\ell}^q$. We set

$$\mathcal{E}_q^+ = \bigoplus_{k+\ell \text{ even}} \mathcal{E}_{k,\ell}^q \quad \text{and} \quad \mathcal{E}_q^- = \bigoplus_{k+\ell \text{ odd}} \mathcal{E}_{k,\ell}^q.$$

Define the flows $\Phi_{t,q}^+$ and $\Phi_{t,q}^-$, acting respectively on the bundles $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$, by

$$\Phi_{t,q}^+ = \bigoplus_{k+\ell \text{ even}} \Phi_t^{k,\ell,q} \quad \text{and} \quad \Phi_{t,q}^- = \bigoplus_{k+\ell \text{ odd}} \Phi_t^{k,\ell,q}.$$

Then $\Phi_{t,q}^\pm$ is a virtual lift of $\widetilde{\varphi}_t$ to the virtual bundles $\mathcal{E}_q^\pm$, in the sense of [Fri95, p. 176]. Also, following [Fri95, p. 176], given a periodic ray γ, one defines $\chi_\gamma(\mathcal{E}_q^\pm) = \chi_\gamma(\mathcal{E}_q^+) - \chi_\gamma(\mathcal{E}_q^-)$. More precisely, given a point $\omega = (z, E) \in G$, $z \in \gamma$, and a bundle $\xi \to G$ over G, one considers the transformation $\Phi_{\tau(\gamma)} : \xi_\omega \to \xi_\omega$, where ξ_ω is the fibre over ω and Φ_t is the lift of the flow $\tilde{\varphi}_t$ to ξ. Then we set $\chi_\gamma(\xi) = \operatorname{tr}\Phi_{\tau(\gamma)}$. For a period ray γ related to a primitive periodic ray $\gamma^\sharp$ one defines $\mu(\gamma) \in \mathbb{N}$ determined by the equality $\tau(\gamma) = \mu(\gamma)\tau(\gamma^\sharp)$.

After this preparation one introduces the zeta function

$$\zeta_q(s) := \exp\Big(-\frac{1}{q}\sum_{\tilde{\gamma}} \frac{\chi_\gamma(\mathcal{E}_q^\pm)}{\mu(\gamma)|\det(\mathrm{Id} - \tilde{P}_\gamma)|}\mathrm{e}^{-s\tau(\gamma)}\Big), \quad \mathrm{Re}(s) \gg 1.$$

This function corresponds exactly to the *flat-trace function* $s \mapsto T^\flat(s)$ introduced by Fried [Fri95, p. 177]. On the other hand, one has

$$\chi_\gamma(\mathcal{E}_q^\pm) = \sum_{k,\ell}(-1)^{k+\ell}\mathrm{tr}\,\Phi^{k,\ell,q}_{\tau(\gamma)}(\omega_{\tilde{\gamma}}).$$

According to the analysis of §3 for the function $\zeta_{\mathrm{N}}(s)$, one deduces that

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta_1(s) = \sum_{\gamma\in\mathcal{P}} \frac{\tau(\gamma^\sharp)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}} = \eta_{\mathrm{N}}(s), \quad \mathrm{Re}\, s \gg 1.$$

Similarly, the argument of §4 implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\log(\zeta_2(s)^2) = 2\sum_{\substack{\gamma\in\mathcal{P}\\ m(\gamma)\in 2\mathbb{N}}} \frac{\tau(\gamma^\sharp)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}} = 2\eta_2(s), \quad \mathrm{Re}\, s \gg 1.$$

Consequently, the representation (4.1) yields

$$\eta_{\mathrm{D}}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\Big(\frac{\zeta_2(s)^2}{\zeta_1(s)}\Big), \quad \mathrm{Re}\, s \gg 1. \tag{5.1}$$

For obstacles with real analytic boundary the flow $\widetilde{\varphi}_t$ is real analytic and the bundles $\mathcal{E}_q^\pm$ are real analytic, too.

For convenience of the reader, we recall the definition of the order of a function f meromorphic on the complex plane (see for instance [Hay64]). For $r \geqslant 0$, denote by $n(r,f)$ the number of poles of f in the disk $\{|z| \leqslant r\}$ counted with their multiplicity. Introduce the (Nevalinna) counting function

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}\mathrm{d}t + n(0,f)\log r.$$

Let $\log^+ : \mathbb{R} \to \mathbb{R}^+$ be the function defined by

$$\log^+ x = \begin{cases} \log x & \text{if} \quad x \geqslant 1, \\ 0 & \text{if} \quad x \leqslant 1. \end{cases}$$

The proximity function $m(r,f)$ is defined by

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r\mathrm{e}^{i\theta})|\mathrm{d}\theta$$

and $T(r,f) = N(r,f) + m(r,f)$ is called the (Nevalinna) characteristic of f. Finally, the order $\rho(f)$ of f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

We are now are in position to apply the principal result of Fried [Fri95, Theorem p. 180] (see also pp. 177–178) saying that the zeta functions $s \mapsto \zeta_k(s)$, $k = 1, 2$, are entire functions with finite orders $\rho(\zeta_k)$. Thus ζ_2^2/ζ_1 is a meromorphic function with order $\max\{\rho(\zeta_1), \rho(\zeta_2)\}$.

Proof of Theorem 3. Denote by $\{\mu_j\} \subset \mathbb{C}$ the set of resonances for the wave equation in the domain $\mathbb{R}^d \setminus D$, with Dirichlet boundary conditions. Our purpose is to prove that there exists $\alpha > 0$ such that

$$\sharp\{\mu_j \ : \ 0 < \operatorname{Im} \mu_j \leqslant \alpha\} = \infty.$$

By the work of Ikawa [Ika88b, Ika90a] and a modification of its proof to cover the case d even (see Appendix B), it is sufficient to show that the Dirichlet series $\eta_{\mathrm{D}}(s)$ cannot be continued as an entire function on $\mathbb{C}$, that is, $\eta_{\mathrm{D}}(s)$ has at least one pole. We proceed by contradiction and assume that $\eta_{\mathrm{D}}(s)$ is an entire function. Applying the representation (5.1), this means that $\zeta_2(s)^2/\zeta_1(s)$ has neither poles nor zeros. As we have mentioned above, this function has finite order, so by the Hadamard factorisation theorem we deduce that $\zeta_2(s)^2/\zeta_1(s) = \exp(Q(s))$ for some polynomial $Q(s)$. This implies that $\eta_{\mathrm{D}}(s) = -Q'(s)$ is a polynomial, which is impossible. Indeed, since $\eta_{\mathrm{D}}(s) \to 0$ as $\operatorname{Re}(s) \to +\infty$, this implies that $Q'(s)$ must be the zero polynomial. By uniqueness of the development of an absolutely convergent Dirichlet series of the form $\sum_n a_n \mathrm{e}^{-\lambda_n s}$ [Per08], this leads to a contradiction. □

Appendix A. Hyperbolicity of the billiard flow

In this appendix we show that the non-grazing flow (ϕ_t) defined in §2.1 is uniformly hyperbolic on the trapped set K_e. Throughout this section we will work with the Euclidian metric. As it was mentioned in §2.4, we can obtain the uniform hyperbolicity of the flow (φ_t) on K in the smooth model from that for (ϕ_t) on K_e. The flow (ϕ_t) is hyperbolic on K_e if for every $z = (x, v) \in \mathring{B} \cap K_e$ we have a splitting

$$T_z\mathbb{R}^d = \mathbb{R}X(z) \oplus E_s(z) \oplus E_u(z),$$

where $X(z) = v$ and $E_s(z)/E_u(z)$ are stable/unstable spaces such that $\mathrm{d}\phi_t(z)$ maps $E_{s/u}(z)$ onto $E_{s/u}(\phi_t(z))$ whenever $\phi_t(z) \in \mathring{B} \cap K_e$, and if for some constants $C > 0, \nu > 0$ independent of $z \in K_e$, we have

$$\|\mathrm{d}\phi_t(z) \cdot v\| \leqslant \begin{cases} C\mathrm{e}^{-\nu t}\|v\|, & v \in E_s(z),\ t \geqslant 0, \\ C\mathrm{e}^{-\nu |t|}\|v\|, & v \in E_u(z),\ t \leqslant 0. \end{cases} \tag{A.1}$$

First, we consider the case of periodic points. Our purpose is to define the unstable and stable manifolds $E_u(z)$ and $E_s(z)$ at a periodic point z, and to estimate the norm of $\mathrm{d}\phi_t(z)|_{E_b(z)}$ for $b = u, s$. Consider a periodic ray γ with reflection points

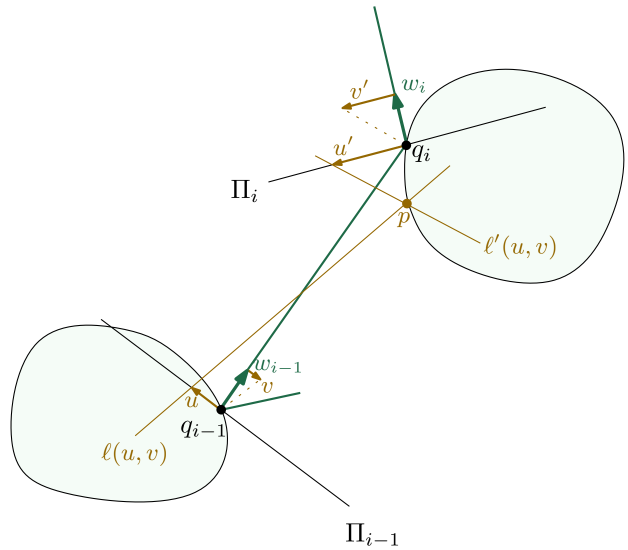

FIGURE 2. The map $\Psi_i : (u, v) \mapsto (u', v')$

$z_i = (q_i, \omega_i)$, $q_i \in \partial D$, $\omega_i \in S^{d-1}$, $i = 0, \dots, m(\gamma) = m$. We will apply the representation of the Poincaré map established in Theorem 2.3.1 and Proposition 2.3.2 in [PS17]. To do this, we recall some notations given in Section 2 of [PS17]. Let $\Pi_i \subset \mathbb{R}^d$ be the plane passing thought q_i and orthogonal to the line $q_i q_{i+1}$ and let Π_i' be the plan passing thought q_i and orthogonal to ω_{i-1}. For $j = i \pmod m$ we set $\Pi_j = \Pi_i$, $q_j = q_i$. Set $\lambda_i = \|q_{i-1} - q_i\|$ and let σ_i be the symmetry with respect to the tangent plane $\alpha_i = T_{q_i}\partial D$. Clearly,

$$\sigma_i(\omega_i) = \omega_{i+1}, \quad \sigma_i(\Pi_i') = \Pi_i, \quad \Pi_0 = \Pi_m.$$

We identify Π_{i-1} and Π_i' by using a translation along the line determined by the segment $[q_{i-1}, q_i]$ and we will write $\sigma_i(\Pi_{i-1}) = \Pi_i$.

We may identify $\Pi_i \times \Pi_i$ with $\Sigma_{z_i} = T_{z_i}(T\mathbb{R}^d)/E_{z_i}$, where E_{z_i} is the two-dimensional space spanned by ω_i and the cone axis at z_i. Given $(u, v) \in \Pi_{i-1} \times \Pi_{i-1}$ sufficiently close to $(0, 0)$, consider the line $\ell(u, v)$ passing through u and having direction $\omega_{i-1} + v$ (the point v is identified with the vector v). Then $\ell(u, v)$ intersects ∂D at a point $p = p(u, v)$ close to q_i. Let $\ell'(u, v)$ be the line symmetric to $\ell(u, v)$ with respect to the tangent plane to ∂D at p and let $u' \in \Pi_i$ be the intersection point of $\ell'(u, v)$ with Π_i. There exists a unique $v' \in \Pi_i$ for which $\omega_i + v'$ has the direction of $\ell'(u, v)$. Thus we get a map

$$\Psi_i : \Pi_{i-1} \times \Pi_{i-1} \ni (u, v) \longmapsto (u', v') \in \Pi_i \times \Pi_i$$

defined for (u, v) in a small neighborhood of $(0, 0)$ (see Figure 2). The smoothness of the billiard ball map $\mathbf{B}$ introduced in §2.4 implies the smoothness of Ψ_i. Next consider the second fundamental form $S(\xi, \eta) = \langle G_i(\xi), \eta \rangle$ for D at q_i, where

$$G_i = \mathrm{d}n_j(q_i) : \alpha_i \longrightarrow \alpha_i$$

is the Gauss map. Introduce a symmetric linear map $\widetilde{\psi}_i$ on Π_i defined by for $\xi, \eta \in \Pi_i'$ by

$$\langle \widetilde{\psi}_i \sigma_i(\xi), \sigma_i(\eta)\rangle = -2\langle \omega_{i-1}, n_j(q_i)\rangle \langle G_i(\pi_i(\xi)), \pi_i(\eta)\rangle,$$

where $\langle .,.\rangle$ denotes the scalar product in $\mathbb{R}^d$ and $\pi_i : \Pi_i' \longrightarrow \alpha_i$ is the projection on α_i along $\mathbb{R}\omega_{i-1}$.

Notice that the non-eclipse condition (1.1) implies that there exists $\beta_0 \in \,]0, \pi/2[$ depending only of D such that for all incoming directions ω_{i-1} and all reflexion points $q_i \in \partial D_j$, one has

$$-\langle \omega_{i-1}, n_j(q_i)\rangle = \langle \omega_i, n_j(q_i)\rangle \geqslant \cos\beta_0 > 0.$$

Consequently, the symmetric map $\widetilde{\psi}_i$ has spectrum included in $[\mu_1, \mu_2]$ with $0 < \mu_1 < \mu_2$ depending only of $\kappa = \cos\beta_0$ and the sectional curvatures of ∂D. Finally, define the symmetric map

$$\psi_i = s_i^{-1} \widetilde{\psi}_i s_i : \, \Pi_m \longrightarrow \Pi_m$$

with $s_i = \sigma_i \circ \sigma_{i-1} \circ \cdots \circ \sigma_1$. By Theorem 2.3.1 in [PS17], the map $\mathrm{d}\Psi_i(0,0)$ has the form

$$\mathrm{d}\Psi_i(0,0) = \begin{pmatrix} I & \lambda_i I \\ \tilde{\psi}_i & I + \lambda_i \tilde{\psi}_i \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix},$$

and the linearized Poincaré map P_γ related to γ is given by

$$P_\gamma = \mathrm{d}(\Psi_m \circ \cdots \circ \Psi_1)(0,0) : \Pi_0 \times \Pi_0 \longrightarrow \Pi_0 \times \Pi_0,$$

which implies

$$P_\gamma = \begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} \begin{pmatrix} I & \lambda_m I \\ \psi_m & I + \lambda_m \psi_m \end{pmatrix} \cdots \begin{pmatrix} I & \lambda_1 I \\ \psi_1 & I + \lambda_1 \psi_1 \end{pmatrix}.$$

Now we repeat without changes the argument of Proposition 2.3.2 in [PS17]. For $k = 0, 1, \dots, m$, consider the space $\mathcal{M}_k$ of linear symmetric non-negative definite maps $M : \, \Pi_k \longrightarrow \Pi_k$. Next, let $\mathcal{M}_k(\varepsilon) \subset \mathcal{M}_k$ be the space of maps such that $M \geqslant \varepsilon I$ with $\varepsilon > 0$. To study the spectrum of P_γ, consider the subspace

$$L_0 = \{(u, M_0 u) : \, u \in \Pi_0\}, \quad M_0 \in \mathcal{M}_0,$$

which is Lagrangian with respect to the natural symplectic structure on $\Pi_0 \times \Pi_0$. By the action of the map $\mathrm{d}\Psi_1(0,0)$, the space L_0 is transformed into

$$L_1 = \{\sigma_1(I + \lambda_1 M_0)u, \sigma_1((I + \lambda_1\psi_1)M_0 + \psi_1)u : \, u \in \Pi_0\} \subset \Pi_1 \times \Pi_1.$$

Introduce the operator

$$\mathcal{A}_i : \mathcal{M}_{i-1} \longrightarrow \mathcal{M}_i$$

defined by

$$\mathcal{A}_i(M) = \sigma_i M (I + \lambda_i M)^{-1} \sigma_i^{-1} + \tilde{\psi}_i.$$

Therefore we may write $L_1 = \{(u, M_1 u) : \, u \in \Pi_1\}$ with $M_1 = \mathcal{A}_1(M_0)$. By recurrence, one defines

$$L_k = \{(u, M_k u) : \, u \in \Pi_k\}, \quad M_k = \mathcal{A}_k(M_{k-1}), \quad k = 1, 2, \dots, m.$$

The maps $\mathcal{A}_k$ are contractions from $\mathcal{M}_{k-1}(\varepsilon)$ to $\mathcal{M}_k(\varepsilon)$, and hence

$$\mathcal{A} = \mathcal{A}_m \circ \cdots \circ \mathcal{A}_1$$

is also a contraction from $\mathcal{M}_0(\varepsilon)$ to $\mathcal{M}_0(\varepsilon)$. We choose $M_0 \in \mathcal{M}_0(\varepsilon)$ as a fixed point of $\mathcal{A}$ and notice that $\varepsilon > 0$ can be chosen uniformly for all periodic rays. Thus we deduce

$$P_\gamma \begin{pmatrix} u \\ M_0 u \end{pmatrix} = \begin{pmatrix} Su \\ M_0 Su \end{pmatrix}$$

with a map $S : \Pi_0 \longrightarrow \Pi_0$ having the form

$$S = \sigma_m(I + \lambda_m \mathcal{A}'_{m-1}(M_0)) \circ \sigma_{m-1}(I + \lambda_{m-1}\mathcal{A}'_{m-2}(M_0)) \circ \cdots \circ \sigma_1(I + \lambda_1 M_0),$$

where $\mathcal{A}'_k = \mathcal{A}_k \circ \mathcal{A}_{k-1} \circ \cdots \circ \mathcal{A}_1$. Setting

$$d_0 = \min_{i \neq j} \operatorname{dist}(D_i, D_j) > 0, \quad d_1 = \max_{i \neq j} \operatorname{dist}(D_i, D_j),$$

and $\beta = \log(1 + \varepsilon d_0)$, one obtains

$$\|Su\| \geqslant (1 + d_0\varepsilon)^m \|u\| = \mathrm{e}^{\beta m}\|u\|.$$

Obviously, the eigenvalues of S are eigenvalues of P_γ and we conclude that P_γ has $(d-1)$ eigenvalues $\nu_1, \ldots, \nu_{d-1}$ satisfying

$$|\nu_j| \geqslant \mathrm{e}^{\beta m}, \quad j = 1, \ldots, d-1.$$

For $0 < \tau < \lambda_1$, consider a point $\rho = \phi_\tau(z) \in \mathring{B} \cap \gamma$, where $z = (x, v) \in \mathcal{D}_{\text{in}}$. The map $\phi_\tau : \mathcal{D}_{\text{in}} \to \mathring{B}$ is smooth near z and moreover $\mathrm{d}\phi_\tau(z) : \Sigma_z \to \Sigma_{\phi_\tau(z)}$. We identify $\Pi_0 \times \Pi_0$ with Σ_z and $\Sigma_{\phi_\tau(z)}$ with the image

$$\mathrm{d}\phi_\tau(z)\Sigma_z = \begin{pmatrix} I & \tau I \\ 0 & I \end{pmatrix} (\Pi_0 \times \Pi_0).$$

Next we define the unstable subspace of $\Sigma_{\phi_\tau(z)}$ as

$$E_u(\phi_\tau(z)) = \mathrm{d}\phi_\tau(z)(L_0) = \begin{pmatrix} I & \tau I \\ 0 & I \end{pmatrix} (L_0).$$

Let $0 < \tau < \lambda_1$, $0 < \sigma < \lambda_{p+1}$ with $p \geqslant 1$ and set $t = -\tau + \sum_{j=1}^p \lambda_j + \sigma$. Then ϕ_t is smooth near ρ and

$$\mathrm{d}\phi_t(\rho)|_{\Sigma_\rho} = \mathrm{d}\phi_\sigma(\mathbf{B}^p(z)) \circ \mathrm{d}\mathbf{B}^p(z) \circ \mathrm{d}\phi_{-\tau}(z) : \Sigma_\rho \to \Sigma_{\phi_t(\rho)}.$$

Thus we have the diagram

$$\begin{array}{ccccccc} & & E_u(\rho) & \xrightarrow{\mathrm{d}\phi_t(\rho)} & E_u(\phi_t(\rho)) & & \\ & & \big\downarrow {\scriptstyle \mathrm{d}\phi_{-\tau}(\rho)} & & \big\uparrow {\scriptstyle \mathrm{d}\phi_\sigma(\mathbf{B}^p(z))} & & \\ \Pi_0 & \xrightarrow{\chi_0} & L_0 & \xrightarrow{\mathrm{d}\mathbf{B}^p(z)} & L_p & \xleftarrow{\chi_p} & \Pi_p, \end{array}$$

where $\chi_0 : \Pi_0 \ni u \mapsto (u, M_0 u) \in L_0 \subset \Pi_0 \times \Pi_0$ and $\chi_p : \Pi_p \ni u \mapsto (u, M_p u) \in L_p \subset \Pi_p \times \Pi_p$. It is easy to obtain an estimate of the action of $\mathrm{d}\phi_t(\rho)|_{E_u(\rho)}$ for $\rho = \phi_\tau(z)$, $v = \mathrm{d}\phi_\tau(z)(u, M_0 u) \in E_u(\rho)$. Clearly,

$$\mathrm{d}\phi_t(\rho) \cdot v = (\mathrm{d}\phi_\sigma(\mathbf{B}^p(z)) \circ \mathrm{d}\mathbf{B}^p(z))(u, M_0 u).$$

By the above argument we deduce

$$\mathrm{d}\mathbf{B}^p(z)(u, M_0 u) = (S_p u, M_p S_p u) \in L_p$$

with

$$S_p = \sigma_p(I + \lambda_p \mathcal{A}'_{p-1}(M_0)) \circ \sigma_{p-1}(I + \lambda_{p-1}\mathcal{A}'_{p-2}(M_0)) \circ \cdots \circ \sigma_1(I + \lambda_1 M_0).$$

Setting $\beta_0 = \beta/d_1$ and $w = (u, M_0 u) = \mathrm{d}\phi_{-\tau}(\rho) \cdot v$, we have

$$\|\mathrm{d}\mathbf{B}^p(z) \cdot w\| = \|(S_p u, M_p S_p u)\| \geqslant \|S_p u\| \geqslant \mathrm{e}^{\frac{\beta}{d_1} p d_1}\|u\| \geqslant \mathrm{e}^{\beta_0(t+\tau-\sigma)}\|u\|,$$

hence we get

$$\|\mathrm{d}\mathbf{B}^p(z) \cdot w\| \leqslant C_0 \mathrm{e}^{-\beta_0 d_1} e^{\beta_0 t}\|w\| = C_0 \mathrm{e}^{-\beta_0 d_1} e^{\beta_0 t}\|\mathrm{d}\phi_{-\tau}(\rho) v\|. \tag{A.2}$$

Here we used the estimate

$$\|w\| = \Big(\|u\|^2 + \|M_0 u\|^2\Big)^{1/2} \leqslant (1 + B_0^2)^{1/2}\|u\|$$

with $\|M_0\|_{\Pi_0 \to \Pi_0} \leqslant B_0$ and we set $C_0 = (1 + B_0^2)^{-1/2}$. The constant B_0 can be chosen uniformly for all M_k and all periodic points since for every non-negative symmetric map M one has

$$\|M(I + \lambda_k M)^{-1}\| \leqslant \frac{1}{\lambda_k} \leqslant \frac{1}{d_0},$$

and the norms $\|\widetilde{\psi}_k\|$ are uniformly bounded by a constant depending on the sectional curvatures and $\kappa > 0$. Consequently,

$$\|\mathcal{A}_k(M)\| \leqslant B_0, \tag{A.3}$$

the same is true for the fixed point $M_0 = \mathcal{A}_m(M_{m-1})$ and the estimate (A.3) is uniform for all periodic points. Finally, estimating the norm of $\mathrm{d}\phi_{-\sigma}(\mathbf{B}^p(z)) = \begin{pmatrix} I & -\sigma I \\ 0 & I \end{pmatrix}$, we obtain $\|\mathrm{d}\phi_\sigma(\mathbf{B}^p(z))\zeta\| \geqslant (1 + d_1)^{-1}\|\zeta\|$ and

$$\begin{aligned} \|\mathrm{d}\phi_t(\rho) v\| &\geqslant (1 + d_1)^{-1} C_0 e^{-\beta_0 d_1} \mathrm{e}^{\beta_0 t}\|\mathrm{d}\phi_{-\tau}(\rho) v\| \\ &\geqslant (1 + d_1)^{-2} C_0 e^{-\beta_0 d_1} \mathrm{e}^{\beta_0 t}\|v\|. \end{aligned}$$

It remains to treat the case $\rho = \phi_\tau(z)$, $z \in \mathcal{D}_{in}$, $0 < t = \tau + \sigma < \lambda_1$. Then $\phi_t(\rho) = \phi_{\tau+\sigma}(z) \in \mathring{B} \cap \gamma$ and we obtain easily an estimate for $\|(\mathrm{d}\phi_{\tau+\sigma}(z)) \cdot v\|$.

Our case is a partial one of a more general setting (see [LW94]) concerning Lagrangian spaces $\{(u, Mu)\}$ with positive definite linear maps M. Such spaces are called positive Lagrangian. A linear symplectic map L is called monotone if it maps positive Lagrangian onto positive Lagrangian. In [LW94] it is proved that any monotone symplectic map is a contraction on the manifold of positive Lagrangian spaces. After a suitable conjugation the map L has the representation (see Proposition 3 in [LW94])

$$L = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} I & R \\ P & I + PR \end{pmatrix}$$

with positive definite matrices P, R. In our situation we have $A = I, R = \lambda_i I$, $P = \psi_i$.

To determine the stable space $E_s(z)$ at z, we will study the flow ϕ_t for $t < 0$ and repeat the above argument leading to a fixed point. The linear map P_γ^{-1} for a periodic ray γ with m reflexions has the representation

$$P_\gamma^{-1} = (\mathrm{d}\Psi_1)^{-1} \circ \cdots \circ (\mathrm{d}\Psi_m)^{-1} : \Pi_0 \times \Pi_0 \longrightarrow \Pi_0 \times \Pi_0,$$

where

$$(\mathrm{d}\Psi_k)^{-1} = \begin{pmatrix} \sigma_k^{-1} & 0 \\ 0 & \sigma_k^{-1} \end{pmatrix} \begin{pmatrix} I + \lambda_k \psi_k & -\lambda_k I \\ -\psi_k & I \end{pmatrix}.$$

Recall that $\Pi_0 = \Pi_m$. Consider a Lagrangian $Q_0 = Q_m = \{(u, -N_m u) : u \in \Pi_0\}$ with a symmetric non-negative definite map $N_m \in \mathcal{M}_0$. Then

$$\begin{aligned}(\mathrm{d}\Psi_m)^{-1} Q_m &= \big\{ \big(\sigma_m^{-1}(I + \lambda_m(\psi_m + N_m))u, -\sigma_m^{-1}(\psi_m + N_m)u\big) \ : \ u \in \Pi_0 \big\} \\ &= \big\{ (u, -N_{m-1}u) : u \in \Pi_{m-1} \big\},\end{aligned}$$

where

$$N_{m-1} = \sigma_m^{-1}(\psi_m + N_m)\Big(I + \lambda_m(\psi_m + N_m)\Big)^{-1} \sigma_m : \Pi_{m-1} \longrightarrow \Pi_{m-1}.$$

By recurrence, introduce the Lagrangian spaces

$$Q_k = \{(u, -N_k u) : u \in \Pi_k\}, \quad N_k = \mathcal{B}_k(N_{k+1}), \quad k = 0, \ldots, m-1,$$

where

$$\mathcal{B}_k(M) = \sigma_{k+1}^{-1}(\psi_{k+1} + M)\Big(I + \lambda_{k+1}(\psi_{k+1} + M)\Big)^{-1} \sigma_{k+1} : \Pi_k \longrightarrow \Pi_k.$$

It is easy to see that $\mathcal{B}_k$ are contractions from $\mathcal{M}_{k+1}(\varepsilon)$ to $\mathcal{M}_k(\varepsilon)$ since

$$\begin{aligned}&\sigma_{k+1}\Big(\mathcal{B}_k(M_1) - \mathcal{B}_k(M_2)\Big)\sigma_{k+1}^{-1} \\ &\qquad = (I + \lambda_{k+1}(\psi_{k+1} + M_1))^{-1}(M_1 - M_2)(I + \lambda_{k+1}(\psi_{k+1} + M_2))^{-1}.\end{aligned}$$

Therefore, $\mathcal{B} = \mathcal{B}_0 \circ \cdots \circ \mathcal{B}_{m-1}$ will be contraction from $\mathcal{M}_0(\varepsilon)$ to $\mathcal{M}_0(\varepsilon)$ and there exists a fixed point $N_m \in \mathcal{M}_0(\varepsilon)$ of $\mathcal{B}$. Moreover,

$$P_\gamma^{-1} \begin{pmatrix} u \\ -N_m u \end{pmatrix} = \begin{pmatrix} \widetilde{S} u \\ -N_m \widetilde{S} u \end{pmatrix}, \quad u \in \Pi_0,$$

where

$$\begin{aligned}\widetilde{S} =& \sigma_1^{-1}(I + \lambda_1(\psi_1 + \mathcal{B}_1'(N_m))) \circ \sigma_2^{-1}(I + \lambda_2(\psi_2 + \mathcal{B}_2'(N_m))) \\ &\qquad \circ \cdots \circ \sigma_m^{-1}(I + \lambda_m(\psi_m + N_m))\end{aligned}$$

and $\mathcal{B}_k' = \mathcal{B}_k \circ \cdots \circ \mathcal{B}_{m-1}$, $k = 1, \ldots, m-1$. Clearly,

$$\|\widetilde{S} u\| \geqslant (1 + d_0 \varepsilon)^m \|u\|, \; u \in \Pi_0,$$

where $\varepsilon > 0$ depends of the sectional curvatures of D. Thus the stable manifold at $\phi_\sigma(z), -\lambda_{m-1} < \sigma < 0$ can be defined as $E_s(\phi_\sigma(z)) = \mathrm{d}\phi_\sigma(z)(Q_m)$ and we may repeat the above argument for the estimate of $\mathrm{d}\phi_t(\phi_\sigma(z))$ acting on $E_s(\phi_\sigma(z))$ for $t < 0$.

The intersection of the unstable and stable manifolds at $y = \phi_t(z)$, $0 < t < \lambda_p$ is $(0, 0)$. Indeed, we have

$$E_u(y) = \mathrm{d}\phi_t(z)(L_{p-1}), \quad E_s(y) = \mathrm{d}\phi_{t-\lambda_p}(\phi_{\lambda_p}(z))(Q_p),$$

where $L_{p-1} = \{(u, M_{p-1}u) : u \in \Pi_{p-1} \times \Pi_{p-1}\}$ and $Q_p = \{(-u, -N_p u) : u \in \Pi_p \times \Pi_p\}$. Assume that $E_u(y) \cap E_s(y) \neq (0,0)$. Then there exists $0 \neq v \in L_{p-1} \cap \mathrm{d}\phi_{-\lambda_p}(\phi_{\lambda_p}(z))(Q_p)$. By the above argument $\mathrm{d}\phi_{-\lambda_p}(\phi_{\lambda_p}(z))(Q_p) = \{(u, -N_{p-1}u) : u \in \Pi_{p-1} \times \Pi_{p-1}\}$. This implies the existence of $u \neq 0$ for which $(M_{p-1} + N_{p-1})u = 0$ which is impossible since $M_{p-1} + N_{p-1}$ is a definite positive map. Consequently, $E_u(y)$ and $E_s(y)$ are transversal subspaces of dimension $d-1$ of Σ_y and we have a direct sum $\Sigma_y = E_u(y) \oplus E_s(y)$.

Now we pass to the estimates of $\mathrm{d}\phi_t(z)|_{E_u(z)}$, where $z \in \mathring{B} \cap K_e$ is not a periodic point. Since $z \in K_e$, the trajectory $\gamma = \{\phi_t(z) : t \in \mathbb{R}\}$ has infinite number successive reflection points $q_k \in \partial D_{i_k}$, $k \in \mathbb{Z}$, with an infinite sequence

$$J_0 = (i_j)_{j \in \mathbb{Z}}, \quad i_j \neq i_{j+1}.$$

For every $p \geqslant p_0 \gg 1$ define the configuration

$$\alpha_p = \begin{cases} (i_{-p}, \ldots, i_0, \ldots, i_p) & \text{if } i_p \neq i_{-p}, \\ (i_{-p}, \ldots, i_0, \ldots, i_{p+1}) & \text{if } i_p = i_{-p}. \end{cases}$$

Repeating α_p infinite times, one obtains an infinite configuration and following the arguments of the proof of Proposition 10.3.2 in [PS17], there exists a periodic ray γ_p following this configuration. Thus we obtain a sequence of periodic rays $(\gamma_{p_0+k})_{k \geqslant 0}$. Let $\{q_{p,k} \in \partial D_{i_k}\}$ be the reflexion points of γ_p. For the periodic ray γ_p passing through $q_{p,0} \in \partial D_{i_0}$ consider the linear space

$$L_{p,0} = \{(u, M_{p,0}u) : u \in \Pi_{p,0}\} \subset \Pi_{p,0} \times \Pi_{p,0}.$$

Our purpose is to show that the symmetric linear maps $M_{p,0} \in \mathcal{M}_{p,0}(\varepsilon)$ composed by some unitary maps converge as $p \to \infty$ to a symmetric linear map $\widetilde{M}_0 \in \mathcal{M}_0(\varepsilon)$ on Π_0. This composition is necessary since the maps $M_{p,0}$, $p \geqslant p_0$, are defined on different spaces. To do this, we will use Lemmas 10.2.1, 10.4.1 and 10.4.2 in [PS17]. Consider the rays γ_{p_0+q}, $q \geqslant 1$, and γ. These rays have reflection points passing successively through the obstacles

$$L' = D_{i_{-p_0-1}}, D_{i_{-p_0}}, \ldots, D_{i_0}, \ldots, D_{i_{p_0}}, D_{i_{p_0+1}} = L''.$$

According to Lemma 10.2.1 in [PS17], there exist uniform constants $C > 0$ and $\delta \in (0,1)$ such that for any $|k| \leqslant p_0$ and $j = 1, \ldots, q$, one has

$$\|q_{p_0+1,k} - q_{p_0+j,k}\| \leqslant C(\delta^{p_0+k} + \delta^{p_0-k}), \quad \|q_{p_0+j,k} - q_k\| \leqslant C(\delta^{p_0+k} + \delta^{p_0-k}).$$

We need to introduce some notations from [PS17, Section 10.4]. Let $x \in \partial D_i$ and $y \in \partial D_j$ with $i \neq j$, and assume that the segment $[x,y]$ is transversal to both ∂D_i and ∂D_j. Let Π be the plane orthogonal to $[x,y]$, passing through x. Let $e = (x-y)/\|x-y\|$, and introduce the projection $\pi : \Pi \longrightarrow T_x(\partial D)$ along the vector e. As above, we define the symmetric linear map $\widetilde{\psi} : \Pi \to \Pi$ by

$$\langle \widetilde{\psi}(u), u \rangle = 2\langle e, n(x) \rangle \langle G_x(\pi(u)), \pi(u) \rangle, \quad u \in \Pi,$$

and notice that

$$\operatorname{spec} \widetilde{\psi} \subset [\mu_1, \mu_2], \quad 0 < \mu_1 < \mu_2.$$

Setting $D_0 = 2C$, we have the estimates

$$\|q_{p_0+j,k} - q_k\| \leqslant D_0 \delta^{p_0+k}, \quad k = -p_0+1, \ldots, 0, \quad j = 1, \ldots, q.$$

Fix $1 \leqslant j \leqslant q$ and introduce the vectors

$$e_k = \frac{q_{k+1} - q_k}{\|q_{k+1} - q_k\|}, \quad e'_k = \frac{q_{p_0+j,k+1} - q_{p_0+j,k}}{\|q_{p_0+j,k+1} - q_{p_0+j,k}\|}.$$

Consider the maps $\widetilde{\psi}_k : \Pi_k \longrightarrow \Pi_k$ and $\widetilde{\psi}'_k : \Pi'_k \longrightarrow \Pi'_k$ related to the segments $[q_{k-1}, q_k]$ and $[q_{p_0+j,k-1}, q_{p_0+j,k}]$, respectively. Let $M_{-p_0+1} : \Pi_{-p_0+1} \longrightarrow \Pi_{-p_0+1}$ and $M'_{-p_0+j} : \Pi'_{-p_0+j} \longrightarrow \Pi'_{-p_0+j}$ be symmetric non-negative definite linear operators. By induction, define

$$M_k = \sigma_k M_{k-1}(I + \lambda_k M_{k-1})^{-1}\sigma_k + \widetilde{\psi}_k, \quad k = -p_0 + 2, \dots, 0,$$

where $\lambda_k = \|q_{k-1} - q_k\|$ and σ_k is the symmetry with respect to $T_{q_k}\partial D$. Similarly, we define M'_k, $k = -p_0 + 2, \dots, 0$, replacing $\widetilde{\psi}_k, \lambda_k$ and σ_k by $\widetilde{\psi}'_k$, $\lambda_{p_0+j,k} = \|q_{p_0+j,k-1} - q_{p_0+j,k}\|$ and σ'_k, respectively. Next, introduce the constants

$$b = (1 + 2\mu_1 \kappa d_0)^{-1} < 1, \quad a_1 = \max\{\delta, b\} < 1,$$

where $d_0 > 0$ and $\kappa > 0$ were defined above. We choose M_{-p_0+1} so that $\|M_{-p_0+1}\| \leqslant B_0$ and by induction one deduces $\|M_k\| \leqslant B_0$. Here $B_0 > 0$ is the constant in (A.3). We have uniform estimates

$$\|M_k\| \leqslant B_0, \quad \|M'_k\| \leqslant B_0, \quad k = -p_0 + 1, \dots, 0. \tag{A.4}$$

Applying in [PS17, Lemma 10.4.1], there exists a linear isometry $A_k : \mathbb{R}^d \to \mathbb{R}^d$ such that $A_k(\Pi'_k) = \Pi_k$, and A_k satisfies the estimates

$$\|A_k - I\| \leqslant C_1 D_0 (1+\delta)\delta^k, \quad \|\widetilde{\psi}_k - A_k \widetilde{\psi}'_k A_k^{-1}\| \leqslant C_2 D_0 (1+\delta)\delta^k, \tag{A.5}$$

for any $k = -p_0 + 1, \dots, 0$. Now we are in position to apply in [PS17, Lemma 10.4.2] saying that with some constant $E > 0$, depending only on D, κ, δ and b, for $k = -p_0 + 1, \dots, 0$ we have

$$\|M_k - A_k M'_k A_k^{-1}\| \leqslant D_0 E a_1^{p_0+k} + b^{2(k+p_0-1)} \|M_{-p_0+1} - A_{-p_0+1} M'_{-p_0+1} A_{-p_0+1}^{-1}\|. \tag{A.6}$$

The norm of the second term on the right hand side is bounded by $2B_0 b^{2(k+p_0-1)}$ and for $k = 0$ one gets

$$\|M_0 - A_0 M'_0 A_0^{-1}\| \leqslant D_0 E a_1^{p_0} + 2B_0 b^{2(p_0-1)}.$$

Applying the above estimate for the rays γ_{p_0+q}, the maps M'_0, A_0 will depend on the ray γ_{p_0+q} and for this reason we denote them by $M'_{q,0}, A_{q,0}$. Now we use these estimates for the maps $M'_{q,0}$, $M_{q',0}$ related to the rays γ_{p_0+q} and $\gamma_{p_0+q'}$ and by the triangle inequality one deduces

$$\left\|A_{q,0} M'_{q,0} A_{q,0}^{-1} - A_{q',0} M'_{q',0} A_{q',0}^{-1}\right\| \leqslant 2D_0 E a_1^{p_0} + 4B_0 b^{2(p_0-1)}. \tag{A.7}$$

Here $A_{q,0}(\Pi'_{q,0}) = \Pi_0$ and $A_{q',0}(\Pi'_{q',0}) = \Pi_0$ are some isometries satisfying the estimates (A.5). Clearly, one obtain a Cauchy sequence $(A_{q,0} M'_{q,0} A_{q,0}^{-1})_{q \geqslant 1}$ which converges to a symmetric non-negative linear map $\widetilde{M_0}$ in Π_0. Moreover, if for every q we have $M'_{q,0} \geqslant \varepsilon I$, then $\widetilde{M_0} \geqslant \varepsilon I$.

After this preparation we define the unstable manifold at $\phi_t(z_0)$ for some $0 < \tau < \|q_1 - q_0\|$ as the subspace

$$E_u(\phi_\tau(z)) = \mathrm{d}\phi_\tau(z)\{(u, \widetilde{M}_0 u) \in \Pi_0 \times \Pi_0 : \ u \in \Pi_0\} \subset \Sigma_{\phi_\tau(z)}.$$

It is important to note that the procedure leading to the estimate (A.6) can be repeated starting with $\widetilde{M}_0$ instead of M_{-p_0+1}. Then if $\widetilde{M}_k$ are the maps obtained from $\widetilde{M}_0$ after successive reflexions, we obtain an estimate

$$\|\widetilde{M}_k - A_k M_k' A_k^{-1}\| \leqslant D_0 E a_1^{p_0+k} + b^{2(k+p_0-1)} \|\widetilde{M}_0 - A_0 \widetilde{M}_0' A_0^{-1}\|$$

for $k = 1, \dots, p_0/2$.

We can repeat the above argument for $\rho = \phi_\tau(z)$, $v \in E_u(\rho)$, and $t = -\tau + \sum_{j=1}^p \lambda_j + \sigma$, where $0 < \tau < \lambda_1$ and $0 < \sigma < \lambda_{p+1}$, to estimate $\|\mathrm{d}\phi_t(\rho) \cdot v\|$. We apply (A) with the expansion map $\widetilde{S}_p$ defined as the composition of the maps $(I + \lambda_k \mathcal{A}_{k-1}'(\widetilde{M}_0))$ and we get (A.4). Finally, the construction of the stable space $E_s(\phi_\sigma(z))$, $-\|q_{-1} - q_0\| < \sigma < 0$ can be obtained by a similar argument and we omit the details.

Appendix B. Ikawa's criterion for even dimensions

In this appendix we prove the result of Ikawa [Ika90a, Theorem 2.1] for all dimensions $d \geqslant 2$. This results is based on Lemma 2.2, Proposition 2.3 and Theorem 2.4 in [Ika90a]. For the modification covering all dimensions $d \geqslant 2$, it is necessary only to modify Lemma 2.2 since the other results are independent of the dimension d. Below we consider only the resonances μ_j for which $0 < \arg \mu_j < \pi$ and we omit this in the notation. In what follows set

$$\Lambda_\rho = \{\mu_j : \ 0 < \operatorname{Im} \mu_j \leqslant \rho |\operatorname{Re} \mu_j|, \ 0 \leqslant \arg \mu_j \leqslant \pi\}.$$

Let $\rho \in C_c^\infty(\mathbb{R}, [0,1])$ be an even function with $\operatorname{supp} \rho \subset]-1, 1[$ such that

$$\rho(t) = 1 \quad \text{if} \quad |t| \ \leqslant 1/2,$$

and with the property that its Fourier transform is nonnegative,

$$\hat{\rho}(k) = \int \mathrm{e}^{itk} \rho(t) \mathrm{d}t \geqslant 0, \quad k \in \mathbb{R}.$$

Let $(\ell_q)_{q \in \mathbb{N}}$ and $(m_q)_{q \in \mathbb{N}}$ be sequences of positive numbers such that $\ell_q \geqslant d_0 > 0$ and $\ell_q, m_q \to \infty$ as $q \to \infty$. Finally, set

$$\rho_q(t) = \rho(m_q(t - \ell_q)), \quad t \in \mathbb{R},$$

and $c_0 = \int \rho(t) \mathrm{d}t$. The result [Ika90a, Lemma 2.2] must be modified as follows.

Lemma B.1. *Let $\alpha \geqslant 1$ and assume that*

$$\#\{j : \mu_j \in \Lambda_\rho^+ : \ 0 < \operatorname{Im} \mu_j \leqslant \alpha\} = P(\alpha) < \infty.$$

Then

$$\sum_{\mu_j \in \Lambda_\rho} |\hat{\rho}_q(\mu_j)| \leqslant C_0 e^\alpha m_q^{d+1} \mathrm{e}^{-\alpha l_q} + c_0 P(\alpha) m_q^{-1} \tag{B.1}$$

for some constant $C_0 > 0$ independent of α and q.

Proof. We write

$$\sum_{z_j\in\Lambda_\rho} = \sum_{\substack{z_j\in\Lambda_\rho\\ \operatorname{Im} z_j>\alpha}} + \sum_{\substack{z_j\in\Lambda_\rho\\ \operatorname{Im} z_j\leqslant\alpha}} = (\mathrm{I}) + (\mathrm{II}).$$

In the sum (II) there is only a finite number of terms and

$$\Big|\int \rho(m_q(t-\ell_q))\mathrm{e}^{it\mu_j}\mathrm{d}t\Big| \leqslant \int \rho(m_q(t-\ell_q))\mathrm{d}t \leqslant c_0 m_q^{-1}.$$

For (I) one integrates by parts,

$$\int \rho(m_q(t-\ell_q))\mathrm{e}^{it\mu_j}\mathrm{d}t = \frac{(-1)^{d+2}m_q^{d+2}}{(i\mu_j)^{d+2}}\int \rho^{(d+2)}(m_q(t-\ell_q))\mathrm{e}^{it\mu_j}\mathrm{d}t. \tag{B.2}$$

Since $\operatorname{supp}\rho_q \subset [\ell_q - m_q^{-1}, \ell_q + m_q^{-1}]$ and $\operatorname{Im}\mu_j > \alpha$, we have

$$|e^{it\mu_j}| \leqslant \mathrm{e}^{-t\operatorname{Im}\mu_j} \leqslant \mathrm{e}^{-\alpha(\ell_q - m_q^{-1})} \leqslant \mathrm{e}^{\alpha}\mathrm{e}^{-\alpha\ell_q}.$$

In particular the right hand side of (B.2) is estimated by $Ce^{\alpha}\frac{e^{-\alpha\ell_q}m_q^{d+1}}{|\mu_j|^{d+2}}\|\rho\|_{C^{d+2}(\mathbb{R})}$ with a constant $C > 0$ independent of j and q. On the other hand, by the results of Vodev [Vod94b], [Vod94a] we have the estimate

$$\#\{\mu_j : |\mu_j| \leqslant k\} \leqslant C_1 k^d,$$

and the series

$$\sum_{|\mu_j|\geqslant 1}\frac{1}{|\mu_j|^{d+2}} = \sum_{k=1}^{\infty}\sum_{k\leqslant|\mu_j|<k+1}\frac{1}{|\mu_j|^{d+2}} \leqslant C_1\sum_{k=1}^{\infty}\frac{(k+1)^d}{k^{d+2}} \leqslant C_2$$

is convergent. This completes the proof. ☐

Notice that the other terms in the trace formula of Zworski (1.2) are easily estimated. In fact, since $\lambda \mapsto \psi(\lambda)$ has compact support, one gets

$$\begin{aligned}\Big|\int\Big(\int \psi(\lambda)\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\cos(\lambda t)\mathrm{d}\lambda\Big)\rho(m_q(t-\ell_q))\mathrm{d}t\Big| &\leqslant C_\psi\int \rho(m_q(t-\ell_q))\mathrm{d}t\\ &\leqslant C_\psi c_0 m_q^{-1}.\end{aligned}$$

Similarly,

$$\Big|\int v_{\rho,\psi}(t)\rho(m_q(t-\ell_q))\mathrm{d}t\Big| \leqslant C_{\rho,\psi}\int \rho(m_q(t-\ell_q))\mathrm{d}t \leqslant c_0 C_{\rho,\psi} m_q^{-1}.$$

We can put the estimates of these terms in $(c_0P(\alpha) + C_3)m_q^{-1}$. Finally, under the assumption of [Ika90a, Lemma 2.2], we have, for all $q \in \mathbb{N}$,

$$|\langle u, \rho_q\rangle| \leqslant C_0\mathrm{e}^{\alpha}m_q^{d+2}\mathrm{e}^{-\alpha l_q} + (c_0P(\alpha) + C_3)m_q^{-1}, \tag{B.3}$$

with constants $C_0, P(\alpha), C_3$ independent of the sequences (ℓ_q) and (m_q).

Define the distribution $\hat{F}_{\rm D} \in \mathcal{D}'(\mathbb{R}^+)$ by

$$\hat{F}_{\rm D}(t) = \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{m(\gamma)} \tau^\sharp(\gamma) \delta(t - \tau(\gamma))}{|\det(I - P_\gamma)|^{1/2}}.$$

Then for $\operatorname{Re} s \gg 1$ we have $\eta_{\rm D}(s) = \langle \hat{F}_{\rm D}(t), {\rm e}^{-st} \rangle$. As we mentioned above, the following results are proved in [Ika90a] and their proofs are independent of the dimension d. For convenience of the reader we present the statements.

Proposition B.2 (Prop. 2.3, [Ika90a]). *Suppose that the function $s \mapsto \eta_{\rm D}(s)$ cannot be prolonged as an entire function of s. Then there exists $\alpha_0 > 0$ such that for any $\beta > \alpha_0$ we can find sequences $(\ell_q), (m_q)$ with $\ell_q \to \infty$ as $q \to \infty$ and such that for all $q \geqslant 0$ one has*

$$e^{\beta \ell_q} \leqslant m_q \leqslant {\rm e}^{2\beta \ell_q} \quad \textit{and} \quad |\langle \hat{F}_{\rm D}, \rho_q \rangle| \geqslant {\rm e}^{-\alpha_0 \ell_q}.$$

Theorem 5 (Theorem 2.4, [Ika90a]). *There are $C, \alpha_1 > 0$ such that for any sequences (ℓ_q) and (m_q), it holds*

$$|\langle u, \rho_q \rangle| \geqslant |\langle \hat{F}_{\rm D}, \rho_q \rangle| - C{\rm e}^{\alpha_1 \ell_q} m_q^{-1}. \tag{B.4}$$

Remark B.3. In [Ika90a, Theorem 2.4], on the right hand side of (B.4), one has the term $m_q^{-\epsilon}$ for some $\epsilon > 0$ instead of m_q^{-1}. In particular the above estimate holds, increasing $\beta > \alpha_0$.

The above theorem is given in [Ika90a] without proof. However its proof repeats that of Proposition 2.2 in [Ika88c] following the procedure described in [Ika85, §3] and exploiting the construction of asymptotic solutions in [Ika88a]. The first term on the right hand side of (B.4) is obtained by the leading term in (1.3) applying the stationary phase argument to a trace of a global parametrix (see Chapter 4 in [PS17]) or to the trace of the asymptotic solutions given below. For the second one we must estimate a sum

$$\sum_{\substack{\gamma \in \mathcal{P} \\ \tau(\gamma) \leqslant \ell_q + m_q^{-1}}} \int_{\ell_q - m_q^{-1}}^{\ell_q + m_q^{-1}} \rho_q(t) r_\gamma(t) {\rm d}t,$$

where r_γ is a function in $L^1_{\rm loc}(\mathbb{R})$, which is obtained from the lower order terms in the application of the stationary phase argument. Since $r_\gamma(t)$ could increase as $t \to \infty$, we need a precise analysis of the behavior of $r_\gamma(t)$.

We discuss briefly the approach of Ikawa and refer to [Ika85], [Ika88a] for more details. First one expresses the distribution $u(t)$ defined in Introduction by the kernels $E(t, x, y)$, $E_0(t, x, y)$ of the operators $\cos(t\sqrt{-\Delta}) \oplus 0$ and $\cos(t\sqrt{-\Delta_0})$, respectively (recall that $-\Delta$ is the Laplacian in $Q = \mathbb{R}^d \setminus D$ with Dirichlet boundary conditions on ∂D). Consider

$$\hat{E}(t, x, y) = \begin{cases} E(t, x, y) & \text{if} \quad (x, y) \in Q \times Q, \\ 0 & \text{if} \quad (x, y) \notin Q \times Q. \end{cases}$$

If $D \subset \{x:\ |x| \leqslant a_0\}$, then

$$\mathrm{supp}_{x,y}\Big(\hat{E}(t,x,y) - E_0(t,x,y)\Big) \subset \big\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d:\ |x| \leqslant a_0 + t,\ |y| \leqslant a_0 + t\big\}.$$

For $t \in \operatorname{supp} \rho_q$ we must study the trace

$$\int_{\Omega_q} \langle \hat{E}(t,x,x) - E_0(t,x,x), \rho_q \rangle \mathrm{d}x$$

with $Q_q = \{x \in Q : |x| \leqslant a_0 + \ell_q + 1\}$. For odd dimensions the kernel $E_0(t,x,x)$ vanishes for $t > 0$. For even dimensions, $x \mapsto E_0(t,x,x)$ is smooth for any $t > 0$ and we can easily estimate

$$\left| \int_{\Omega_q} \langle E_0(t,x,x), \rho_q \rangle \mathrm{d}x \right| \leqslant A_0 m_q^{-1}$$

with $A_0 > 0$ independent of q by using the representation of the kernel $E_0(t,x,y)$ by oscillatory integrals with phases $e^{ik(\langle x-y,\omega\rangle \pm t)}$ (see for example, [PS17, §3.1]).

Now, choose $g \in C_c^\infty(Q_q)$ and write the kernel $E(t,x,y)g(y)$ of $\cos(t\sqrt{-\Delta})g(y)$ as

$$E(t,x,y)g(y) = (2\pi)^{-d} \int_{\mathbb{S}^{d-1}} d\omega \int_0^\infty k^{d-1} u(t,x;k,\omega) e^{-ik\langle y,\omega\rangle} g(y) dk,$$

where $u(t,x;k,\omega)$ is the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \quad \text{in} \quad \mathbb{R} \times Q, \\ u = 0 \quad \text{on} \quad \mathbb{R} \times \partial Q, \\ u(0,x) = \widetilde{g}(x)\mathrm{e}^{ik\langle x,\omega\rangle}, \quad \partial_t u(0,x) = 0, \end{cases}$$

with a function $\widetilde{g} \in C_c^\infty(Q)$ equal to 1 on $\operatorname{supp} g$. In the works [Ika85, Ika88a, Ika88c, Bur93]) of Ikawa and Burq, asymptotic solution $w^{(N)} = w_{q,+}^{(N)} + w_{q,-}^{(N)}$ of the above problem have been constructed. They have the form

$$w_{q,\pm}^{(N)}(t,x;k,\omega) = \sum_{|\mathbf{j}|d_0 \leqslant a_0 + \ell_q + 1} \mathrm{e}^{ik(\varphi_{\mathbf{j}}^{\pm}(x,\omega) \mp t)} \sum_{h=0}^{N} v_{\mathbf{j},h}^{\pm}(t,x,\omega)(ik)^{-h}.$$

Here $\mathbf{j} = \{j_1, j_2, \ldots, j_n\}$, $j_k \in (1, \ldots, r)$, $j_k \neq j_{k+1}$, $k = 1, 2, \ldots, n-1$, $|\mathbf{j}| = n$ is a configuration related to the rays reflecting successively on $\partial D_{j_1}, \partial D_{j_2}, \ldots, \partial D_{j_n}$ (see §2.3). The phases $\varphi_{\mathbf{j}}$ are constructed successively starting from $\langle x, \omega\rangle$ and following the reflections on obstacles determined by the configuration $\mathbf{j}$. The amplitudes are determined by transport equations. The reader may consult [Ika85, §3], [Ika88c, Equations (3.2) and (3.3)], [Ika88a, §4] and [Bur93] for the construction of $v_{\mathbf{j}}^{\pm}$. The function $u - w^{(N)}$ is solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)(u - w^{(N)}) = k^{-N} F_N(t,x;k,\omega) \quad \text{in} \quad \mathbb{R} \times Q, \\ u - w^{(N)} = k^{-N} b_N(t,x;k,\omega) \quad \text{on} \quad \mathbb{R} \times \partial Q, \\ (u - w^{(N)})(0,x;k,\omega) = \partial_t(u - w^{(N)})(0,x;k,\omega) = 0. \end{cases}$$

Here F_N is obtained as the action of $(\partial_t^2 - \Delta_x)$ to the amplitudes $v^{\pm}_{\mathbf{j},N}$, while b_N is obtained by the traces on ∂Q of the amplitudes $v_{\mathbf{j},N}$. It is important to note that the asymptotic solutions $w^{(N)}$ are independent of the sequence (m_q). The integral involving $u - w^{(N)}$ is easily estimated and it yields term $\mathcal{O}(m_q^{-1})$ (see [Ika85]). For the integral with $w^{(N)}_{q,\pm}$ involving $w^{(N)}_{q,\pm}$ one applies the stationary phase argument as $k \to \infty$ for the integration with respect to $x \in Q_q$, $\omega \in S^{d-1}$, considering t as a parameter. Next, in [Ika88a], estimates of the p derivatives of $v^{\pm}_{\mathbf{j},h}(x,t,\omega)$ with respect to $x \in Q_q$, $\omega \in S^{d-1}$ with bound $C_p e^{-\alpha_2 \ell_q}(t+1)^h$ have been established. Here $C_p > 0$ and $\alpha_2 > 0$ are independent of ℓ_q. This implies the estimate

$$\big|\langle u - \hat{F}_{\mathrm{D}}, \rho_q\rangle\big| \leqslant A\mathrm{e}^{-\alpha_2 \ell_q} \#\{\mathbf{j} : |\mathbf{j}| \leqslant \frac{2\ell_q}{d_0}\}\ell_q^{2N+2} m_q^{-1}$$

with constant $A > 0$ independent of q. Since

$$\#\{\mathbf{j} : |\mathbf{j}| \leqslant \frac{2\ell_q}{d_0}\} \leqslant \mathrm{e}^{\alpha_3 \ell_q}, \ \forall q,$$

we obtain (B.4), by using a partition on unity $\sum_j \psi_j(x) = 1$ on Q_q.

Combining Proposition B.2 and the estimates (B.3) and (B.4), it is easy to find $\alpha > 0$ so that the strip $\{z : \ 0 \leqslant \operatorname{Im} z \leqslant \alpha\}$ contains an infinite number of resonances. Indeed, let

$$\alpha = (d+2)(\alpha_0 + \alpha_1 + 1), \quad \beta = \frac{\alpha}{d+2},$$

and suppose $P(\alpha) < \infty$. Then

$$m_q^{d+1}\mathrm{e}^{-\alpha \ell_q} \geqslant \mathrm{e}^{(d+1)\beta \ell_q}\mathrm{e}^{-\alpha \ell_q} = \mathrm{e}^{-\beta \ell_q}$$

and from (B.3), (B.4) one deduces

$$(C_0 \mathrm{e}^{\alpha} + c_0 P(\alpha))\mathrm{e}^{-\beta \ell_q} \geqslant \mathrm{e}^{-\alpha_0 \ell_q} - C\mathrm{e}^{\alpha_1 \ell_q}\mathrm{e}^{-\beta \ell_q} = \mathrm{e}^{-\alpha_0 \ell_q}(1 - C\mathrm{e}^{-\ell_q}).$$

Since $\beta > \alpha_0$, letting $q \to \infty$ yields a contradiction.

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