ALGANT Master Thesis in Mathematics

# Local Picture of Twisted Curves

with an Introduction to the Theory of Deligne-Mumford Stacks

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*A mio fratello,*

# Contents

# Introduction

A *nodal curve* $C$ over a field $k$ is, roughly speaking, a curve whose non-smooth points are ordinary double points. The classical way to describe families of such curves over a base scheme $S$ is through a flat morphism $f : C \to S$, whose fibers are nodal curves over the residue field. It is possible to endow nodal curves, over a base scheme $S$, with the action of a group of roots of unity $\mu_r$. In the simplest case of the curve $Spec\left(\frac{k[x,y]}{(xy)}\right)$ over the field $k$, a root of unity $\xi$ acts by sending $x \to \xi x$ and $y \to \xi^{-1}y$.
A *twisted curve* over a base scheme $S$ is a nodal curve which aquires an *orbifold* structure at its nodes, through the action of $\mu_r$. To be more precise, it is a Deligne-Mumford stack $\mathcal{C} \to S$, which étale locally looks like the stack-theoretic quotient $[C/\mu_r]$, where $C$ is a nodal curve with an action of a group of roots of unity $\mu_r$.
Twisted curves were introduced, for the first time, by Abramovich and Vistoli in [3] in 2002; in this paper, they study the space of curves of fixed genus and degree on a proper Deligne-Mumford stack $\mathcal{M} \to S$. It turns out that it is not possible to give a compactification of this space directly mimicking the approach Kontsevich used to compactify the space of curves, of fixed genus and degree, over a projective scheme $M \to S$, namely adding stable curves; the right objects to add are indeed *stable twisted curves.*
In my work, I prove a characterization of twisted curves $\mathcal{C} \to S$ in terms of their geometric fibers, under the assumption of excellence of the base scheme $S$. In particular, this characterization relies on the étale local description of the curves at their nodes, and gives an important result yet for the case of nodal curves.
The thesis is divided into three chapters. As the title suggests, the first two chapters are devoted to an introduction to the theory of Deligne-Mumford stacks, reflecting the presentation given by Olsson in [14], while in the third one I present my original work.
In the first chapter, we establish the categorical setting to describe *stacks.* Given a *site* $\mathcal{C}$, namely a category with a *Grothendieck topology*, a *stack* $\mathcal{F} \to \mathcal{C}$ is a category fibered in groupoids over $\mathcal{C}$ in which "we can glue morphisms and objects". More precisely, for every objects $x, x' \in \mathcal{F}(U)$, for $U \in \mathcal{C}$, the functor $\underline{Isom}(x, x')$ is a sheaf and every *descent datum* is effective. This means that if we are given a covering $\{U_i \to U\}_i$ of $U \in \mathcal{C}$ and objects $x_i \in \mathcal{F}(U_i)$, with transition isomorphisms on the intersections satisfying the cocycle condition, there exists an object $x \in F(U)$ whose restrictions through the covering are the $x_i$.
In the second chapter, we introduce *group schemes* and their actions on schemes. In general, it is not possible to have a nice notion of *quotient* of a scheme $X$ by the action of a group scheme $G$ in the category of schemes. Having this motivation in mind, we enlarge the category of schemes to the category of *algebraic spaces*; these objects are nicely characterized as sheaf-theoretic quotients of a scheme by an *étale equivalence relation.* In this way, it is possible to take the quotient of a scheme by

the free action of a group scheme. However, even if these objects describe nicely the quotients, they lose important information, namely the original action of the group; a natural generalization that takes that into account are *algebraic stacks.* In particular, we focus on a class of such objects, namely *Deligne-Mumford stacks*; these stacks are étale locally isomorphic to the stack-theoretic quotient $[V/G]$, with $V$ a scheme and $G$ a finite group.
In the third chapter, we give a precise definition of when a scheme *looks like* another scheme étale locally at a geometric point; we say that two schemes $X, Y$ have the same *local picture* at geometric points $\overline{x}, \overline{y}$ if $X^{sh} \cong Y^{sh}$, where the strict henselizations are taken with respect to $\overline{x}, \overline{y}$. In the same flavour, it is possible to define when two morphisms $C \to S$ and $C' \to S'$ have the same local picture at geometric points. From this point of view, we define a *nodal curve* as a morphism $C \to S$ whose local picture at a geometric point is given by:

$$Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right) \to Spec(A)$$

with $\alpha \in A$. My first characterization shows how, under the hypothesis of excellence of the base scheme $S$, it is equivalent, for a flat morphism locally of finite presentation $C \to S$, to be a nodal curve and for all the geometric fibers to be nodal curves. Secondly, I generalize this characterization for twisted curves. Provided a similar notion of local picture for algebraic stacks, a *twisted curve* is a morphism $\mathcal{C} \to S$, with $\mathcal{C}$ a tame Deligne-Mumford stack, whose local picture at a geometric point is given by:

$$\left[Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)/\mu_r\right] \to Spec(A)$$

The final result shows again, under suitable hypotheses, that $\mathcal{C} \to S$ is a twisted curve if and only if all its geometric fibers are twisted curves.

## Acknowledgements

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To conclude, I thank my family for supporting me in every choice I made and Chiara, who was always by my side.

# Chapter 1

# Stacks

## 1.1 Grothendieck Topologies

### 1.1.1 Sites

**Definition 1.1.** Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is the assignment to each object $U$ of $\mathcal{C}$ of a collection of sets of arrows $\{U_i \to U\}_i$, called *coverings* of $U$, such that the following conditions are satisfied:

(i) If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering.

(ii) If $\{U_i \to U\}_i$ is a covering and $V \to U$ is any arrow, then the fiber products $U_i \times_U V$ exist and the collection of projections $\{U_i \times_U V \to V\}_i$ is a covering.

(iii) If $\{U_i \to U\}_i$ is a covering, and for each index $i$ we have a covering $\{V_{ij} \to U_i\}_j$, then the collection of composites $\{V_{ij} \to U_i \to U\}_{ij}$ is a covering of $U$.

A category $\mathcal{C}$ with a Grothendieck topology is called a *site*.

**Example 1.2** (classic topology)**.** Let $X$ be a topological space and denote by $I_X$ the category of open subsets of $X$ with inclusions as arrows. A covering of an open subset $U \subseteq X$ is an open covering $\{U_i \to U\}_i$ in the classical sense. Note that this gives a Grothendieck topology on $I_X$, since in $I_X$ the fiber product is $U_i \times_U U_j = U_i \cap U_j$.

**Example 1.3** (Localized site)**.** Let $\mathcal{C}$ be a site and let $X \in \mathcal{C}$. Define the category $\mathcal{C}/X$, whose objects are arrows $f : Y \to X$, with $Y \in \mathcal{C}$ and morphisms between $f : Y \to X$ and $f' : Y' \to X$ are morphisms $h : Y \to Y'$ such that $f = f' \circ h$. We may refer to these morphisms as $X$-*morphisms*. $\mathcal{C}/X$ has a natural structure of site. For any object $f : Y \to X$, a covering is a collection $\{\tilde{f}_i : (f_i : Y_i \to X) \to (f : Y \to X)\}_i$ such that $\{\tilde{f}_i : Y_i \to Y\}_i$ is a covering of $Y$ in $\mathcal{C}$.

We will focus our attention on categories of schemes; denote by $Sch$ the category of schemes with morphisms of schemes as arrows, and for every scheme $S$, denote by $Sch/S$ the category of $S$-schemes $T \to S$ with $S$-morphisms as arrows. In particular, given schemes $X, T$, we will denote by $X(T) = Hom_{Sch/S}(T, X)$ the set of arrows from $T$ to $X$.

**Example 1.4** (Small étale site)**.** Let $S$ be a scheme. Define $Et(S)$ to be the full subcategory of $Sch/S$ whose objects are étale morphisms $T \to S$; a covering of $T \to S$ is a collection $\{T_i \to T\}_i$ such that:

$$\bigsqcup_i T_i \to T$$

is surjective. This assignment gives a Grothendieck topology, since being étale is invariant by base change.

**Example 1.5** (Big étale site)**.** Let $S$ be a scheme. We define on $Sch/S$ a Grothendieck topology, such that for any $S$-scheme $T \to S$ a covering is a collection $\{T_i \to T\}_i$ such that $T_i \to T$ is an étale morphism and:

$$\bigsqcup_i T_i \to T$$

is surjective.

**Example 1.6** (fppf site)**.** Let $S$ be a scheme. We define on $Sch/S$ a Grothendieck topology, such that for any $S$-scheme $T \to S$ a covering is a collection $\{T_i \to T\}_i$ such that $T_i \to T$ is a flat morphism locally of finite presentation and:

$$\bigsqcup_i T_i \to T$$

is surjective; here fppf stands for *fidèlement plat et de preséntation finie.*

*Remark* 1.7. We introduced a relative notion of Grothendieck topology on the category of schemes, but we can get an "absolute" notion of these topologies taking just $S = \mathbb{Z}$, so $Sch = Sch/\mathbb{Z}$.

It is in general more convenient to use the notion of the "big site" of a scheme, on which the functorial behaviour of a scheme becomes more clear, as we will explain later.

### 1.1.2 Sheaves on Sites

**Definition 1.8.** Let $\mathcal{C}$ be a site. A *presheaf* on $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to Set$.
A presheaf is called a *sheaf* if for every object $U \in \mathcal{C}$ and covering $\{U_i \to U\}_{i \in I}$, the sequence:

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j)$$

is exact, where the last two maps are induced by the natural projections. Notice that the exactness of the sequence means that the first map is an equalizer of the two maps induced by the projections.

*Remark* 1.9. Notice that for a topological space $X$, we can recover the usual definition of a (pre)sheaf. In fact, consider the Grothendieck topology described in Example 1.2 on $I_X$. Then a presheaf is a functor $F : I_X^{op} \to Set$ and it is a sheaf if for every open $U \subseteq X$ and covering $\{U_i \to U\}_{i \in I}$ of $U$, the sequence:

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j)$$

is exact.

As promised before, schemes present naturally a functorial behaviour, but we have to recall first of all the Yoneda Lemma.

Let $\mathcal{C}$ be a category. For every $X \in \mathcal{C}$, define the functor:

$$h_X : \mathcal{C}^{op} \to Set$$

by $h_X(Y) = Hom_{\mathcal{C}}(Y, X)$ for every $Y \in \mathcal{C}$ and for any morphism $f : Z \to Y$, $h_X(f) : Hom_{\mathcal{C}}(Y, X) \to Hom_{\mathcal{C}}(Z, X)$ which sends $g \mapsto g \circ f$.
We will refer to these functors $h_X$ as *functor of points.*

**Lemma 1.10** (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, and denote by $Hom(\mathcal{C}^{op}, Set)$ the category of functors $\mathcal{C}^{op} \to Set$. Then the functor:*

$$\mathcal{C} \to Hom(\mathcal{C}^{op}, Set)$$

*which sends $X \mapsto h_X$ is fully faithful.*

We will refer to this embedding of categories as the *Yoneda embedding.* This lemma is of great importance; first of all, being fully faithful means that for every $X, Y \in \mathcal{C}$, then:

$$Hom_{\mathcal{C}}(X, Y) \cong Hom_{\widehat{\mathcal{C}}}(h_X, h_Y)$$

where $\widehat{\mathcal{C}} = Hom(\mathcal{C}^{op}, Set)$. So, in the case of $\mathcal{C} = Sch$, we can describe schemes and their morphisms just looking at their functors of points. Moreover, notice that in general this functor fails to be essentially surjective; indeed, the essential image of the functor is made of all the representable functors $\mathcal{C}^{op} \to Set$. This means that in general the two categories are not equivalent; so, in the case of $Sch$, this embedding enlarges the usual category of schemes, which will allow us in the future to introduce objects generalizing schemes.

## 1.2 Fibered Categories

We develop here the basic formalism about fibered categories; in particular, focus is given to categories fibered in groupoids, the standard setting in which stacks are introduced. Proofs and a more complete discussion can be found in Vistoli [16] and Olsson [14].

**Definition 1.11.** Let $\mathcal{C}$ be a category.

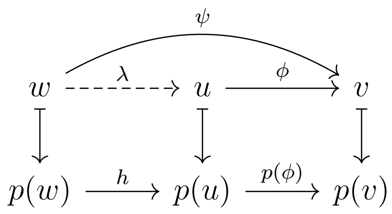
(i) A *category over* $\mathcal{C}$ is a pair $(\mathcal{F}, p)$ where $\mathcal{F}$ is a category and $p : \mathcal{F} \to \mathcal{C}$ is a functor.

(ii) A morphism $\phi : u \to v$ in $\mathcal{F}$ is called *cartesian* if for any other object $w \in \mathcal{F}$ and morphism $\psi : w \to v$ with factorization:

$$p(w) \xrightarrow{h} p(u) \xrightarrow{p(\phi)} p(v)$$

of $p(\psi)$, there exists a unique morphism $\lambda : w \to u$ such that $\phi \circ \lambda = \psi$ and $p(\lambda) = h$. In particular, if $\phi : u \to v$ is a cartesian arrow, then $u$ is called a *pullback* of $v$ along $p(\phi)$.

We can find motivation on this terminology by the following picture:




which shows how $u$ actually "acts" as a pullback of $v$ along $p(\phi)$.

*Remark* 1.12. It is clear by the universal property imposed that the pullback is unique, up to unique isomorphism.

For a category $p : \mathcal{F} \to \mathcal{C}$ over $\mathcal{C}$ and an object $U \in \mathcal{C}$, we denote by $\mathcal{F}(U)$ the *fiber* of $p$ in $U$ the subcategory of $\mathcal{F}$ consisting of objects $u$ such that $p(u) = U$ and morphisms $\phi : u' \to u$ such that $p(\phi) = id_U$.

**Definition 1.13.** Let $\mathcal{C}$ be a category.

(i) a *fibered category* over $\mathcal{C}$ is a category $p : \mathcal{F} \to \mathcal{C}$ over $\mathcal{C}$ such that for every morphism $f : U \to V$ in $\mathcal{C}$ and $v \in \mathcal{F}(V)$ there exists a cartesian morphism $\phi : u \to v$ such that $p(\phi) = f$ (so in particular $u \in \mathcal{F}(U)$).

(ii) Let $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$, $p_{\mathcal{G}} : \mathcal{G} \to \mathcal{C}$ be fibered categories over $\mathcal{C}$. A *morphism of fibered categories* $\mathcal{F} \to \mathcal{G}$ is a functor $g : \mathcal{F} \to \mathcal{G}$ such that:

(a) $p_{\mathcal{G}} \circ g = p_{\mathcal{F}}$,

(b) $g$ sends cartesian morphisms to cartesian morphisms.

**Example 1.14.** Let $\mathcal{C}$ be a site. Then any presheaf:

$$\mathcal{F} : \mathcal{C}^{op} \to Set$$

gives rise to a fibered category:

$$p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$$

where the fibers are the set $\mathcal{F}(U)$, and the pullbacks exist by the definition of presheaf.

*Remark* 1.15. Often, fibered categories are seen as *presheaves with values in categories*. To explain this point of view, given a fibered category $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$, consider the functor:

$$\overline{p_{\mathcal{F}}} : \mathcal{C}^{op} \to Cat$$

That sends $U \to \mathcal{F}(U)$. The axioms for a fibered category, namely the existence and compatibility of pullbacks, mimic the behaviour of presheaves of a site with values in sets. However, we have to be careful: *Cat*, the category of categories, is not a category, but a *2-category*. A 2-category is a collection of objects, with morphisms between objects (*1-morphisms*) and morphisms between morphisms (*2-morphisms*), with axioms similar to the theory of categories. *Cat* is naturally a 2-category, where the objects are categories, 1-morphisms are functors and 2-morphisms are natural transformations of functors. Hence, from this point of view, fibered categories are the natural generalization of presheaves on a site, as stacks will be the natural generalization of sheaves.

### 1.2.1 Categories Fibered in Groupoids

In the theory of stacks, it is more interesting to look at a particular class of fibered categories, namely the *categories fibered in groupoids*.

**Definition 1.16.** Let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category over $\mathcal{C}$.

(i) We say that $\mathcal{F}$ is a *category fibered in groupoids* over $\mathcal{C}$ if for every $U \in \mathcal{C}$, $\mathcal{F}(U)$ is a groupoid.

(ii) We say that $\mathcal{F}$ is a *category fibered in sets* over $\mathcal{C}$ if for every $U \in \mathcal{C}$, $\mathcal{F}(U)$ is a *set*, namely a groupoid where the only morphisms are the identities.

**Example 1.17.** Let $X$ be a scheme. Consider the functor:

$$p : Sch/X \to Sch$$

defined by $p(f : Y \to X) = Y$ and for every $X$-morphism $h : Y \to Y'$, $p(h) = h$. $p : Sch/X \to Sch$ is a fibered category. In fact, let $h : Y \to Y'$ be a morphism of schemes, and $g : Y' \to X$ an object lying over $Y$. Then $g \circ h : Y \to X$ is the pullback required. Notice in particular that $p : Sch/X \to Sch$ is a category fibered in sets; consider a scheme $Y$ and the fiber $Sch/X(Y)$. For any two objects $f : Y \to X$ and $g : Y \to X$, there exists a morphism $h : Y \to Y$ lying above the identity if and only if $f = g$, in which case $h = id_Y$.

**Example 1.18** (Isom functor)**.** We describe now an important construction on a category fibered in groupoids. Let $\mathcal{C}$ be a category and $p : \mathcal{F} \to \mathcal{C}$ a category fibered in groupoids. For any $X \in \mathcal{C}$ and $x, x' \in \mathcal{F}(X)$, define the presheaf:

$$\underline{Isom}(x, x') : (\mathcal{C}/X)^{op} \to Set$$

as follows. For any morphism $f : Y \to X$, choose pullbacks $f^*x$ and $f^*x'$, and set:

$$\underline{Isom}(x, x')(f : Y \to X) = Isom_{\mathcal{F}(Y)}(f^*x, f^*x')$$

For a composition:

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

the pullback $(fg)^*x$ is a pullback along $g$ of $f^*x$ and therefore there is a canonical map:

$$g^* : \underline{Isom}(x, x')(f : Y \to X) \to \underline{Isom}(x, x')(fg : Z \to X)$$

compatible with composition.
In particular, if $x = x'$, we get a presheaf of groups:

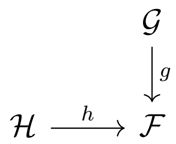
$$\underline{Aut}_x : (\mathcal{C}/X)^{op} \to Group$$

*Remark* 1.19. Up to canonical isomorphism the presheaf $\underline{Isom}(x, x')$ is independent of the choice of pullbacks.

### 1.2.2 Fiber Products

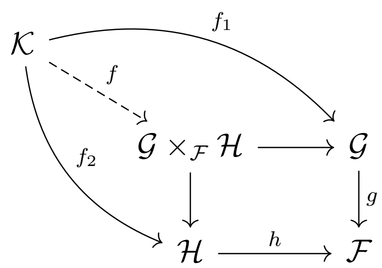
To conclude this section, we define the fiber product of categories fibered in groupoids.

Let $\mathcal{C}$ be a category and:




a diagram of categories fibered in groupoids over $\mathcal{C}$. Define a category fibered in groupoids $\mathcal{G} \times_{\mathcal{F}} \mathcal{H}$ as follows. Its objects are triples $(u, v, \alpha)$ where $u \in \mathcal{G}, v \in \mathcal{H}$ and $\alpha : g(u) \to f(v)$ is a morphism in a fiber of $\mathcal{F}$; a morphism $(u, v, \alpha) \to (u', v', \alpha')$ is a pair of morphisms $(\beta_1 : u \to u', \beta_2 : v \to v')$ in the fibers of $\mathcal{G}, \mathcal{H}$ such that $h(\beta_2) \circ \alpha = \alpha' \circ g(\beta_1)$. Moreover, there is natural morphism $\pi_1 : \mathcal{G} \times_{\mathcal{F}} \mathcal{H} \to \mathcal{G}$ mapping $(u, v, \alpha)$ to $u$ and $(\beta_1, \beta_2)$ to $\beta_1$, and similarly a morphism $\pi_2 : \mathcal{G} \times_{\mathcal{F}} \mathcal{H} \to \mathcal{H}$.

We call $\mathcal{G} \times_{\mathcal{F}} \mathcal{H}$ the *fiber product* of $\mathcal{G}, \mathcal{H}$ along $g, h$. We can see that the fiber product of categories fibered in groupoids comes with an universal property. If $\mathcal{K}$ is a category fibered in groupoids over $\mathcal{C}$, and $f_1 : \mathcal{K} \to \mathcal{G}$, $f_2 : \mathcal{K} \to \mathcal{H}$ are two morphisms of categories fibered in groupoids such that $g \circ f_1 \cong h \circ f_2$, then there exists a unique morphism: $f : \mathcal{K} \to \mathcal{G} \times_{\mathcal{F}} \mathcal{H}$ up to isomorphism of morphisms, such that $f_1 \cong \pi_1 \circ f$ and $f_2 \cong \pi_2 \circ f$. Pictorially:




Be careful that the diagram is not commutative in the usual sense, but up to isomorphism.

*Remark* 1.20. As explained in Remark 1.15, 2-categories give a better environment where to study fibered categories. Hence, from the point of view of 2-categories, we have just defined the *2-categorical fiber product* of two objects. Keep attention that when talking about 2-categories, two morphisms could be not equal, but just isomorphic.

## 1.3 Stacks

Let $\mathcal{C}$ be a site and $\mathcal{F} \to \mathcal{C}$ a category fibered in groupoids over $\mathcal{C}$. Let $U \in \mathcal{C}$ and let $\{U_i \to U\}$ be a covering of $U$. Set $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$ for each triple of indices $i, j, k$.

**Definition 1.21.** Let $U \in \mathcal{C}$ and let $\{\sigma_i : U_i \to U\}_i$ be a covering of $U$; denote by $\sigma_{ij} : U_{ij} \to U$ the induced morphisms. A *descent datum* for $U$ is a collection of objects $\xi_i \in \mathcal{F}(U_i)$, with isomorphisms $\phi_{ij} : p_i^* \xi_i \to p_j^* \xi_j$ in $\mathcal{F}(U_{ij})$, such that the following cocyle condition holds:

$$p_{ik}^* \phi_{ik} = p_{ij}^* \phi_{ij} \circ p_{jk}^* \phi_{jk}$$

for any triple of indices $i, j, k$, where $p_i, p_{ij}$ are the natural projections. We call the isomorphisms $\phi_{ij}$ *transition isomorphisms* of the descent data.
We say that the descent data is *effective* if there exists $\xi \in \mathcal{F}(U)$ and isomorphisms $\phi_i : \sigma_i^* \xi \to \xi_i$ such that $\sigma_{ij}^* \phi_j = \phi_{ij} \circ \sigma_{ij}^* \phi_i$.

*Remark* 1.22. In this definition, we made the implicit choice of a pullback object for any projection; nevertheless, it is possible to give a definition of a descent data completely free of the choice of pullbacks.

**Definition 1.23.** Let $\mathcal{F} \to \mathcal{C}$ be a category fibered in groupoids over a site $\mathcal{C}$. We say that $\mathcal{F}$ is a *stack* if:

(i) For any $U \in \mathcal{C}$ and $x, x' \in \mathcal{F}(U)$, the functor:

$$\underline{Isom}(x, x') : (\mathcal{C}/U)^{op} \to Set$$

is a sheaf on $\mathcal{C}/U$,

(ii) every descent datum is effective.

*Remark* 1.24. With this definition, we can see a stack as a category where we can glue the objects of the category through the effectiveness of the descent data and, for any two objects, we can glue morphisms between them.

Let's see, first of all, why stacks can be seen as objects generalizing schemes.

**Theorem 1.25.** *Let $S$ be a scheme. For any scheme $X$ over $S$, the functor of points $h_X$ is a sheaf for the fppf topology on $Sch/S$.*

*Proof.* See [14] Theorem 4.1.2. □

*Remark* 1.26. Since the fppf topology is finer than the étale topology on $Sch/S$, we get that $h_X$ is a sheaf also for the étale topology.

**Proposition 1.27.** *A scheme $X$ over $S$ is a stack in the fppf site $Sch/S$.*

*Proof.* The condition on the $\underline{Isom}$ functor is trivial since a scheme gives rise to a category fibered in sets. By Theorem 1.25 $h_X$ is a sheaf in the fppf site, and this gives the effectiveness of the descent data. □

To conclude, let $\mathcal{F} \to \mathcal{C}$ and $\mathcal{G} \to \mathcal{C}$ be categories fibered in groupoids over $\mathcal{C}$. We will denote by $HOM_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ the category of morphisms of categories fibered in groupoids between $\mathcal{F}$ and $\mathcal{G}$. We already discussed that fibered categories are objects generalizing presheaves (as *presheaves in categories*); in the same fashion, stacks are objects generalizing sheaves on sites (as *sheaves in categories*). From this point of view, starting from a category fibered in groupoids, we can construct an associated stack, which we will call *stackification*, following the same analogy of the sheafification of a presheaf.

**Theorem 1.28** (Stackification). *Let $p : \mathcal{F} \to \mathcal{C}$ be a category fibered in groupoids. Then there exists a stack $\mathcal{F}^a$ over $\mathcal{C}$ and a morphism of fibered categories $q : \mathcal{F} \to \mathcal{F}^a$ such that for any stack $\mathcal{G}$ over $\mathcal{C}$ the induced functor:*

$$HOM_{\mathcal{C}}(\mathcal{F}^a, \mathcal{G}) \to HOM_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$$

*is an equivalence of categories.*

*Proof.* See Olsson [14] Theorem 4.6.5. □

# Chapter 2

# Algebraic Stacks

In this chapter, we are interested in studying the action of a *group scheme* $G$ on a scheme $X$. There are various notions of quotient of the action we can introduce; we will focus on the *stack quotient*, which will be the prototype of a *Deligne-Mumford stack*.

## 2.1 Group Schemes

**Definition 2.1.** Let $G$ be a scheme over $S$. We say that $(G, m_G, i_G, e_G)$ is a *group scheme over* $S$ if $m_G : G \times_S G \to G$, $i_G : G \to G$, $e_G : S \to G$ are morphisms such that the following diagrams are commutative:

(i) identity:

$$\begin{array}{ccc} S \times_S G & \xrightarrow{e_G \times id_G} & G \times_S G \\ & \searrow & \downarrow{\scriptstyle m_G} \\ & & G \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G \times_S S & \xrightarrow{id_G \times e_G} & G \times_S G \\ & \searrow & \downarrow{\scriptstyle m_G} \\ & & G \end{array}$$

(ii) associativity of multiplication:

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{m_G \times id_G} & G \times_S G \\ \downarrow{\scriptstyle id_G \times m_G} & & \downarrow{\scriptstyle m_G} \\ G \times_S G & \xrightarrow{m_G} & G \end{array}$$

(iii) inverse:

$$\begin{array}{ccc} G & \xrightarrow{i_G \times id_G} & G \times_S G \\ \downarrow & & \downarrow{\scriptstyle m_G} \\ S & \xrightarrow{e_G} & G \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G & \xrightarrow{id_G \times i_G} & G \times_S G \\ \downarrow & & \downarrow{\scriptstyle m_G} \\ S & \xrightarrow{e_G} & G \end{array}$$

We call $e_G$ the *identity* , $i_G$ the *inverse* and $m_G$ the *multiplication* of $G$.

*Remark* 2.2. Equivalently, we can define group schemes functorially. Indeed, a scheme $G$ is a group scheme if it comes with a functor $p : Sch/S \to Group$ that factorizes with the forgetful functor:

$$\mathcal{C}^{op} \xrightarrow{p} Group \xrightarrow{for} Set, \qquad h_G = for \circ p : \mathcal{C}^{op} \to Set$$

where $h_G$ is the functor of points of $G$. By Yoneda Lemma, this is the same as saying that $G$ is a group scheme if and only if for any $U \in Sch/S$, $h_G(U)$ has a group structure and for any morphism $U \to V$, the induced morphism $h_G(V) \to h_G(U)$ is a group homomorphism.

Given two group schemes $G, H$, a *homomorphism* $G \to H$ is a morphism $f : G \to H$ such that the diagram:

$$\begin{array}{ccc} G \times_S G & \xrightarrow{m_G} & G \\ \downarrow{\scriptstyle f \times f} & & \downarrow{\scriptstyle f} \\ H \times_S H & \xrightarrow{m_H} & H \end{array}$$

is commutative. Equivalently, $G \to H$ is an homomorphism if and only if for any scheme $U$, $h_G(U) \to h_H(U)$ is a group homomorphism.

### 2.1.1 Group Scheme Actions

The main interest is not the study of group schemes, but rather the study of the *group action* of a group scheme on another scheme. We are going to introduce here just the *left* action of a group scheme, but we can provide naturally similar arguments for right actions.

**Definition 2.3.** Let $G$ be a group scheme over $S$ and $X$ a scheme over $S$. A *left action of $G$ on $X$* is a morphism $\alpha : G \times X \to X$ such that the following diagrams are commutative:

(i) action of the identity:

$$\begin{array}{ccc} S \times_S X & \xrightarrow{e_G \times id_X} & G \times_S X \\ & \searrow & \downarrow{\scriptstyle \alpha} \\ & & X \end{array}$$

(ii) associativity of the action with respect to multiplication:

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{m_G \times id_X} & G \times_S X \\ \downarrow{\scriptstyle id_G \times \alpha} & & \downarrow{\scriptstyle \alpha} \\ G \times_S X & \xrightarrow{\alpha} & G \end{array}$$

*Remark* 2.4. As before, we can translate this condition in the categorical language: a left action of $G$ on $X$ is a natural transformation $h_G \times h_X \to h_X$ such that for any scheme $U$, $h_G(U) \times h_X(U) \to h_X(U)$ is a left action of $h_G(U)$ on $h_X(U)$.

**Example 2.5** (Discrete group schemes)**.** Let $S$ be a scheme. A *discrete group scheme* is a group scheme which is a *discrete object* of the category $Sch/S$, namely an object of the form $\bigsqcup_{i\in I} S$, with $I$ a family of indices. Discrete group schemes are essential to describe the action of classical groups in the scheme-theoretic sense. Given a scheme $X$, an action of a discrete group scheme $G$ on $X$ is a morphism of the form:

$$G \times_S X \to X$$

Since $G$ is discrete, it follows that $G \times_S X = \bigsqcup_{i\in I} X$, hence giving the action on $X$ is equivalent to give morphisms:

$$\alpha_i : X \to X$$

that satisfy the axioms of an action. We can translate the action in the functorial sense: given a scheme $T$ over $S$, we want to describe:

$$(G \times_S X)(T) \to X(T)$$

or equivalently:

$$\left(\bigsqcup_{i\in I} X\right)(T) \to X(T)$$

Notice that in general:

$$\left(\bigsqcup_{i\in I} X\right)(T) \neq \bigsqcup_{i\in I} X(T)$$

Indeed, the equality fails as soon as $T$ is not connected.

**Example 2.6.** Let $X = Spec\left(\frac{k[x,y]}{(xy)}\right) \to Spec(k)$, with $k$ an algebraic closed field such that $char(k) \neq 2$. Set $G = Spec(k) \sqcup Spec(k)$. We endow $X$ of an action of $G$; to give the map:

$$G \times X \to X$$

is equivalent to give the map:

$$X \sqcup X \to X$$

which is the data of two maps $\alpha_i : X \to X$. We take $\alpha_1 = id_X$ and $\alpha_2 : X \to X$ the map that on the global sections:

$$\frac{k[x,y]}{(xy)} \to \frac{k[x,y]}{(xy)}$$

sends $x \mapsto -x$ and $y \mapsto -y$. We can translate the action in the functorial sense: given a connected scheme $T$ over $Spec(k)$, the action:

$$G(T) \times X(T) \to X(T)$$

is equivalent, as before, to:

$$X(T) \sqcup X(T) \to X(T)$$

given by two maps $X(T) \to X(T)$. Recall that:

$$X(T) = Hom_{Spec(k)}(T, X) \cong Hom_k\left(\frac{k[x,y]}{(xy)}, \mathcal{O}(T)\right)$$

Thus, the elements of $X(T)$ are the data of two elements $a, b \in \mathcal{O}_T(T)$ such that $ab = 0$. Finally, the two maps are the identity and the map that sends the morphism $(x \mapsto a, y \mapsto b)$ to the morphism $(x \mapsto -a, y \mapsto -b)$.
Geometrically, we described two incident affine lines with an involution on the lines.

**Example 2.7.** This is a generalization of the Example 2.6. In the same notation, just set $G = \bigsqcup_{i=1}^{n} Spec(k)$, with $char(k)$ not dividing $n$. Fix also a primitive $n$-root of unity $\xi$. Then the action of $G$ on $X$ is given by:

$$\bigsqcup_{i=1}^{n} X \to X$$

where the map $\alpha_i$ is given by, on the global sections:

$$\frac{k[x,y]}{(xy)} \to \frac{k[x,y]}{(xy)}$$

sending $x \mapsto \xi^i x$ and $y \mapsto \xi^{-i} y$.
In this example, $G$ acts as a group of roots of unity on two incident lines, and this example will be of great importance in the development of twisted curves.

We want to describe the relation between two schemes $X, Y$, both endowed with the action of a group scheme $G$.

**Definition 2.8.** Let $X, Y$ be schemes with the action of a group scheme $G$ over $S$. We say that a morphism $\pi : X \to Y$ is $G$-*equivariant* if the diagram:

$$\begin{CD} G \times_S X @>>> X \\ @V{id_G \times f}VV @VV{f}V \\ G \times_S Y @>>> Y \end{CD}$$

is commutative.

Equivariant morphisms will play a crucial role in the construction of the stack-theoretic quotient of the action of a group scheme on a scheme.

### 2.1.2 Quotients of Finite Group Actions

The rest of this chapter will be devoted to build a suitable setting for the existence and nice behaviour of *quotients* of schemes by a group scheme action.

Let's start with a basic but important case. Consider the action of a finite group $G$ on a ring $A$. This translates, in the language of schemes, into an action of the discrete group scheme $G$ on $Spec(A)$ (as in Example 2.5). Hence, we can consider the subring of $A$ of $G$-invariants element, namely:

$$A^G = \{a \in A : g \cdot a = a \text{ for all } g \in G\}$$

The inclusion $A^G \to A$ induces the map $\pi : Spec(A) \to Spec(A^G)$.

**Definition 2.9.** Let $X$ be a scheme and $G$ be a group scheme acting on it. We say that $(Y, \pi)$ is a *categorical quotient* of $X$ by the action of $G$ if $\pi : X \to Y$ is a morphism of schemes such that:

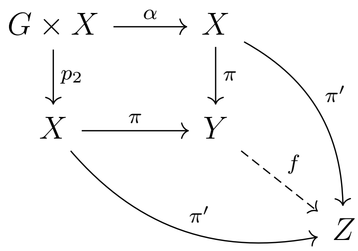
(i) invariance:

$$\begin{CD} G \times X @>{\alpha}>> X \\ @V{p_2}VV @VV{\pi}V \\ X @>{\pi}>> Y \end{CD}$$

is commutative, with $\alpha$ the action and $p_2$ the natural projection and $Y$ a scheme,

(ii) universal property:
For any scheme $Z$ with a map $\pi' : X \to Z$ satisfying the commutative diagram

as above, there exists a unique map $f : Y \to Z$ such that:




is commutative.

From this point of view, we have that the morphism $\pi : Spec(A) \to Spec(A^G)$ described before is a categorical quotient.

**Theorem 2.10.** *Let $S$ be a scheme, $Spec(A)$ be an affine scheme over $S$. If $G$ is a finite group acting on $A$, then:*

$$Spec(A) \to Spec(A^G)$$

*is a categorical quotient for the induced action of $G$ on $Spec(A)$.*

*Proof.* It is a particular case of [14] Theorem 6.2.2. □

*Remark* 2.11. We introduced the notion of categorical quotient for the category of schemes. Nevertheless, we could generalize it, taking into consideration categories enlarging the category of schemes, for the example working with *algebraic spaces* (see for example [6], V, Théorème 4.1).

Let's see some examples of how to compute these quotients.

**Example 2.12.** In the notation of Example 2.7, we endowed $X$ of an action of a finite group $G$. Hence we want to describe $\left(\frac{k[x,y]}{(xy)}\right)^G$. Indeed, consider the map:

$$\frac{k[u,v]}{(uv)} \to \frac{k[x,y]}{(xy)}$$

sending $u \mapsto x^r$ and $v \mapsto y^r$, where $char(k)$ does not divide $r$. Then it is well defined and the map presents in fact $Spec\left(\frac{k[u,v]}{(uv)}\right)$ as the categorical quotient of $X$ by the action of $G$. Notice, in particular, that the quotient is again isomorphic to $X$ (but the map on the quotient is clearly not an isomorphism if $r \geq 1$).

## 2.2 Algebraic Spaces

Unfortunately, the case we did in the last section, namely affine schemes with the free action of a finite group, was very restrictive. Let's see how we can widen our class of examples.

### 2.2.1 Étale Equivalence Relations

**Definition 2.13.** Let $S$ be a base scheme and $X$ an $S$-scheme. An *equivalence relation* on $X$ is a monomorphism of schemes:

$$R \hookrightarrow X \times_S X$$

such that for every $S$-scheme $T$:

$$R(T) \subseteq X(T) \times X(T)$$

is an equivalence relation on $X(T)$.
Moreover, we say that it is an *étale equivalence relation* if the two maps $R \to X$ induced by the natural projections are étale.

**Example 2.14.** Equivalence relations generalize free action of group schemes. In fact, let $G$ be a group scheme acting on a scheme $X$ and denote by $\rho : G \times X \to X$ the action. Then we define:

$$j : G \times X \to X \times X$$

given by $(g, x) \mapsto (x, \rho(g, x))$. If the action is free, namely the action of $G(T)$ on $X(T)$ is free for every scheme $T$, then $j$ is an equivalence relation on $X$. Suppose moreover that $G$ is a discrete group scheme. Then $j$ is an étale equivalence relation. Indeed, étaleness of a morphism is a local property and locally the compositions of $j$ with the projections are isomorphisms.

Consider now $Sch/S$ as a site endowed of the étale topology. We introduce another notion of quotient; let $R$ be an étale equivalence relation on a scheme $X$ over $S$. Consider the presheaf:

$$(Sch/S)^{op} \to Set, \quad T \mapsto X(T)/R(T)$$

In general this functor fails to be a sheaf for the étale topology. Denote by $X/R$ the associated sheaf. This object provides a better notion of quotient of a scheme, as can always be constructed starting from an étale equivalence relations, but in general is not represented by a scheme.
In particular, if the étale equivalence relation $R$ on $X$ is induced by the free action of a group scheme, we will denote the quotient as $X/G$.

### 2.2.2 Representable Morphisms

We focus now on the category of sheaves (on the étale topology) on $Sch/S$, with $S$ a scheme. Indeed, as we remarked in Chapter 1, using Yoneda Lemma we can embed the category $Sch/S$ into its category of presheaves, and by Theorem 1.25 we know that schemes are sheaves for the étale topology. In this way, we have a larger category to construct quotients, as the ones of the form $X/R$, with $X$ a scheme and $R$ an étale equivalence relation. In order to characterize these quotients, we introduce the notion of *representable morphism*.

**Definition 2.15.** Let $S$ be a scheme and $f : F \to G$ a morphism of sheaves on $Sch/S$ with the étale topology.

(i) $f$ is *representable by schemes* if for every $S$-scheme $T$ and morphism $T \to G$ the fiber product $F \times_G T$ is a scheme.

(ii) Let $P$ be a stable property of morphisms of schemes. If $f$ is representable by schemes, $f$ has the property $P$ if for every $S$-scheme $T$ the morphism $F \times_G T \to T$ has the property $P$.

*Remark* 2.16. Let $\mathcal{C}$ be a site. A property $P$ of a morphism $X \to Y$ is *stable* if for every covering $\{Y_i \to Y\}_i$, the induced maps $X \times_Y Y_i \to Y_i$ have the property $P$ if and only if $X \to Y$ has the property $P$.

*Remark* 2.17. As recalled before, by the Yoneda embedding we can enlarge the category of schemes, hence we will use the notation $T$ to denote both the scheme and its functor of points $h_T$. From this point of view, schemes and morphisms of schemes are thought in a functorial way.

**Example 2.18.** If $F, G$ are representable sheaves, then any morphism $f : F \to G$ is representable by schemes. Indeed, if $F \cong h_X, G \cong h_Y$, for any $g : T \to Y$, we have that:

$$F \times_G T \cong h_X \times_{h_Y} h_T \cong h_{X \times_Y T}$$

**Lemma 2.19.** *Let $S$ be a scheme and $F$ be a sheaf on $Sch/S$ with the étale topology. Suppose that the diagonal morphism $\Delta : F \to F \times F$ is representable by schemes. Then if $T$ is a scheme, any morphism $f : T \to F$ is representable by schemes.*

*Proof.* Let $T'$ be a scheme and $g : T' \to F$ be a morphism. Then:

$$T \times_F T' \cong T \times_F T' \times_{F \times F} F \times F \cong T \times T' \times_{F \times F} F$$

which is represented by a scheme by the hypothesis on the diagonal. □

### 2.2.3 Characterization of Algebraic Spaces

**Definition 2.20.** Let $S$ be a scheme. An *algebraic space* over $S$ is a functor:

$$X : (Sch/S)^{op} \to Set$$

such that:

(i) $X$ is a sheaf for the étale topology,

(ii) the diagonal $\Delta : X \to X \times X$ is representable by schemes,

(iii) there exists an $S$-scheme $U$ with a surjective étale morphism $U \to X$.

*Remark* 2.21. Notice that condition (iii) makes sense since the morphism $U \to X$ is representable by schemes by Lemma 2.19.

**Example 2.22.** Every scheme is an algebraic space; by Theorem 1.25 schemes are sheaves for the étale topology, every morphism of schemes is representable by schemes and the identity provides the étale surjection. Thus, the category of algebraic spaces enlarges the category of schemes.

Algebraic spaces are motivated by their nice characterization in terms of the quotient sheaves of étale equivalence relations, as we can see by the Proposition.

**Proposition 2.23.** *Let $S$ be a scheme. Then:*

*(a) Let $X$ be a scheme and $R$ an étale equivalence relation on $X$ over $S$. Then $X/R$ is an algebraic space.*

*(b) Let $Y$ be an algebraic space over $S$, $X \to Y$ an étale surjective morphism with $X$ a scheme. Then:*

  *(i) $R = X \times_Y X$ is a scheme,*
  *(ii) $R \hookrightarrow X \times_S X$ is an étale equivalence relation,*
  *(iii) the natural map $X/R \to Y$ is an isomorphism.*

*Proof.* See [14] Olsson, Proposition 5.2.5. □

**Corollary 2.24.** *Let $S$ be a scheme, $Y : (Sch/S)^{op} \to Set$ a functor. Then the following are equivalent:*

*(a) $Y$ is an algebraic space,*

*(b) $Y \cong X/R$, for a scheme $X$ and an étale equivalence relation $R$ on $X$.*

*Proof.* Follows by Proposition 2.23. □

Let's see some example of algebraic spaces. In particular, we will notice why sometimes they fail to be representable by a scheme.

**Example 2.25.** As in Example 2.14, if $G$ is a discrete group scheme acting freely on a scheme $X$, then we obtain an induced étale equivalence relation $R$ on $X$. We denote by $X/G$ the resulting algebraic space.

**Example 2.26.** Let $k$ be a field and consider the scheme:

$$U = Spec\left(\frac{k[x,y]}{(xy)}\right)$$

obtained by gluing two copies of the affine line along the origin. Let $U' \subset U$ be the open subset obtained by removing the origin. Set:

$$R = U \sqcup U'$$

and consider the two maps:

$$\pi_i : R \to U$$

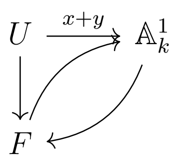
for $i = 1, 2$ defined as follows. The restriction of both $\pi_1, \pi_2$ to $U$ is the identity. On $U'$, we define $\pi_1$ to be the natural inclusion and $\pi_2$ to be the map which switches the two components. Then the resulting map:

$$\pi_1 \times \pi_2 : R \to U \times U$$

is an étale equivalence relation. Let $F = U/R$ be the resulting algebraic space; $F$ is not a scheme. Indeed, the map:

$$x + y : U \to \mathbb{A}^1_k$$

is easily checked to be universal in the category of ringed spaces for maps from $U$ which factor through $F$ (see [14] Exercise 6.J). The map $x + y$ is invariant, hence induces a map $F \to \mathbb{A}^1_k$. But, if $F$ were a scheme, then $U \to F$ would factorize in a map $\mathbb{A}^1_k \to F$, as shown in the diagram:




thus the map $F \to \mathbb{A}^1_k$ would be an isomorphism. But this is not possible, as the map $U \to \mathbb{A}^1_k$ is not étale.

## 2.3 Algebraic Stacks

So far, in this chapter we have not mentioned stacks yet. We saw that using algebraic spaces, we can take quotients of étale equivalence relations on schemes, working on the category of sheaves on schemes. However, in Chapter 1, we gave a description of stacks as *sheaves of categories*; this point of view will be useful to introduce a quotient that takes into consideration not only the class of equivalence of the objects, but also the morphisms between them.

### 2.3.1 Representable Morphisms

We start as before with the notion of *representable morphism*, as we did for algebraic spaces.

**Definition 2.27.** Let $S$ be a scheme and $f : \mathcal{X} \to \mathcal{Y}$ a morphism of stacks over $S$.

(i) $f$ is *representable* if for every $S$-scheme $U$ and morphism $U \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} U$ is an algebraic space.

(ii) Let $P$ be a stable property of morphisms of algebraic spaces. If $f$ is representable, $f$ has the property $P$ if for every $S$-scheme $U$ the morphism $\mathcal{X} \times_{\mathcal{Y}} U \to U$ has the property $P$.

**Lemma 2.28.** *If a morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is representable, then for every algebraic space $V$ and morphism $V \to \mathcal{Y}$, the fiber product $V \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.*

*Proof.* See Olsson [14] Lemma 8.1.3. □

**Lemma 2.29.** *Let $S$ be a scheme and $\mathcal{X}$ be a stack over $S$. Suppose that the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable. Then if $T$ is a scheme, any morphism $f : T \to \mathcal{X}$ is representable.*

*Proof.* Let $T'$ be a scheme and $g : T' \to \mathcal{X}$ be a morphism. Then:

$$T \times_{\mathcal{X}} T' \cong T \times_{\mathcal{X}} T' \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} \times \mathcal{X} \cong T \times T' \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$$

which is represented by an algebraic space by the hypothesis on the diagonal. □

As we did for algebraic spaces, we describe the properties of morphism of stacks in a relative way, via representability. In particular, the representability of the diagonal will be crucial in the next definition.

### 2.3.2 Deligne-Mumford Stacks

**Definition 2.30.** A stack $\mathcal{X}$ over $S$ is called an *algebraic stack* if:

(i) the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable,

(ii) there exists a smooth surjective morphism $\pi : U \to \mathcal{X}$, with $U$ a scheme.

*Remark* 2.31. Notice that condition (ii) makes sense since the morphism $U \to \mathcal{X}$ is representable by Lemma 2.29.

We can see that an algebraic stack is almost a generalization of an algebraic space: we asked the same requirement about the representability of the diagonal, and the existence of the surjection of a scheme into our object (in this case a *smooth* surjection). The only difference now is that $\mathcal{X}$ is a category fibered in groupoids, an object encoding more information than a sheaf of sets. In the next definition, we introduce the main object of study of this chapter.

**Definition 2.32.** An algebraic stack $\mathcal{X}$ over $S$ is called a *Deligne-Mumford stack* if there is an étale surjective morphism $\pi : U \to \mathcal{X}$, with $U$ a scheme.

In the next section we will see why Deligne-Mumford stacks are nice objects to describe quotients of schemes by the action of finite groups, and how to characterize them.

## 2.4 Stacky Quotients

Given a scheme $X$ with the free action of a group $G$ over $S$, we saw that we can make a quotient in a natural way inducing an equivalence relation $R = G \times X \to X \times X$ and then sheafifying the functor (for the étale topology):

$$(Sch/S)^{op} \to Set, \quad T \mapsto X(T)/R(T)$$

to get the sheaf we denoted $X/G$. We would like to generalize this construction for stacks. From the functoriality of the group action, we know that for any $S$-scheme $T$, $X(T)$ has an action of $G(T)$.

Consider so the following category:

$$[X(T)/R(T)]$$

The objects are the elements of $X(T)$, and for any elements $x, y \in X(T)$, a morphism $x \to y$ is an element $g \in G(T)$ such that $g \cdot x = y$.

In this way, we do not lose the information of the action while describing this quotient. Unfortunately, this is not a stack in general (for the étale topology), as much as the presheaf $T \mapsto X(T)/R(T)$ was not a sheaf. Hence, we denote by:

$$[X/G]$$

the associated stack to the category fibered in groupoids which is given by:

$$(Sch/S)^{op} \to Groupoids, \quad T \mapsto [X(T)/R(T)]$$

We are going to give now an explicit description of this stacky quotients, in terms of *$G$-torsors*.

*Remark* 2.33. We are mostly interested in the action of a group scheme $G$ on a scheme $X$. Nevertheless, we can generalize this construction, and assume that $X$ is an algebraic space, with the action of a group scheme; we can naturally introduce the structure of (étale) site on the category of algebraic spaces and proceed in an analogous way. See [14] Example 8.1.12 for such constructions.

**Definition 2.34.** Let $G$ a group scheme over $S$ and $Y$ be an $S$-scheme.

(i) A *trivial $G$-torsor* over $Y$ is a scheme $\mathcal{P}$ with an action of $G$ with a $G$-invariant map $\mathcal{P} \to Y$ such that there is a $G$-equivariant isomorphism $\phi : G \times Y \cong \mathcal{P}$ such that the diagram:

$$\begin{array}{ccc} G \times Y & \xrightarrow{\phi} & \mathcal{P} \\ & \searrow^{pr_2} & \downarrow f \\ & & Y \end{array}$$

is commutative, where the action of $G$ on $G \times Y$ is given by:

$$G \times G \times Y \to G \times Y, \quad (h, g, y) \mapsto (hg, y)$$

(ii) An *étale $G$-torsor* over $Y$ is a scheme $\mathcal{P}$ with an action of $G$ with a $G$-invariant map $\mathcal{P} \to Y$ such that there is an étale covering $\{Y_i \to Y\}_i$ such that for every $i$, $\mathcal{P} \times_Y Y_i$ is a trivial $G \times_Y Y_i$-torsor on $Y_i$.

*Remark* 2.35. We could give, in general, the definition of torsor for any site, just replacing the étale coverings of an object with the appropriate coverings of the site. In this work, since we are just interested in the étale topology, we will omit the adjective *étale.*

With the notion of torsor, we can give an equivalent definition of the stack $[X/G]$.

**Definition 2.36.** Let $X$ be a scheme and $G$ a group scheme acting on $X$ over $S$. Denote by $[X/G]$ the category fibered in groupoids whose objects are triples $(T, \mathcal{P}, \pi)$ where:

(i) $T$ is an $S$-scheme,

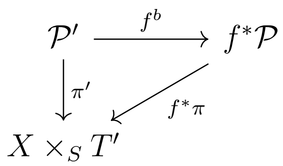
(ii) $\mathcal{P}$ is a $G_T = G \times_S T$-torsor over $T$,

(iii) $\pi : \mathcal{P} \to X \times_S T$ is a $G_T$-equivariant morphism on $Sch/T$.

A morphism:

$$(T', \mathcal{P}', \pi') \to (T, \mathcal{P}, \pi)$$

is a pair $(f, f^b)$ where $f : T' \to T$ is an $S$-morphism of schemes and $f^b : \mathcal{P} \to f^*\mathcal{P}$ is an isomorphism of $G_{T'}$-torsors, such that the induced diagram:



is commutative.

**Proposition 2.37.** $[X/G]$ *is a stack.*

*Proof.* [14] Theorem 4.2.12 on descent of sheaves on a site implies that every descent datum is effective. □

If we assume a nicer structure on $G$, also the quotient will be better behaved.

**Proposition 2.38.** *If $G$ is a smooth group scheme acting on $X$, then $[X/G]$ is an algebraic stack.*

*Proof.* [14] Example 8.1.12. □

*Remark* 2.39. Going through the proof, we see that a smooth surjection that presents the quotient as an algebraic stack is:

$$X \to [X/G]$$

which, seen as an element of $[X/G]\,(X)$, corresponds to the trivial torsor $G \times X$ over $X$ with the $G$-equivariant map $G \times X \to X$ given by the action of $G$ on $X$.

Nevertheless, these two definition of stacky quotient are the same, which allows us to use the more explicit one through the notion of torsors.

**Proposition 2.40.** *Let $X$ be a scheme with the action of a group scheme $G$ over $S$. Then $[X/G]$ (as defined in 2.36) is the stackification of the category fibered in groupoids given by:*

$$(Sch/S)^{op} \to Groupoids, \quad T \mapsto [X(T)/R(T)]$$

*Proof.* See [15] Stacks Project Tag 04WM. □

We conclude revisiting some examples of group actions, taking the quotients in the stack-theoretic sense.

**Example 2.41.** Consider the setting of Example 2.7. Recall that $G = \bigsqcup_{i=1}^{n} Spec(k)$ is a finite group acting on $X = Spec\left(\frac{k[x,y]}{(xy)}\right)$ over $Spec(k)$ as a group of roots of unity (hence a cyclic group) and $char(k)$ does not divide $n$. Consider the quotient stack:

$$\left[Spec\left(\frac{k[x,y]}{(xy)}\right)/G\right]$$

We want to describe the (category of) $\overline{k}$-points of the quotient. Notice that all the étale torsors over an algebraically closed (or even separable closed) field are trivial, since for a field $k$, étale $k$-algebras are finite separable extensions of $k$. Hence different points correspond, up to isomorphism (the trivialization of the torsor) to different equivariant maps on the trivial torsor. The trivial torsor is:

$$G_{\overline{k}} = G \times_{Spec(k)} Spec(\overline{k}) = \bigsqcup_{i=1}^{n} Spec(\overline{k})$$

Equivariant maps $f : G_{\overline{k}} \to X_{\overline{k}}$ have to make the following diagrams:

$$\begin{array}{ccc} G_{\overline{k}} & \xrightarrow{f} & X_{\overline{k}} \\ \downarrow{\scriptstyle g\cdot} & & \downarrow{\scriptstyle g\cdot} \\ G_{\overline{k}} & \xrightarrow{f} & X_{\overline{k}} \end{array}$$

commutative for every $g \in G_{\overline{k}}$; by $g\cdot$ we mean the action of an element $g \in G_{\overline{k}}$. To give the map $G_{\overline{k}} \to X_{\overline{k}}$ is to give $n$ points of $X_{\overline{k}}$. However, if we fix a generator $\gamma \in G_{\overline{k}}$ and we send it to an element $x \in X_{\overline{k}}$, the compatibility of the diagram means that we have to send $\gamma^2 \mapsto \gamma \cdot x$ and in general $\gamma^i \mapsto \gamma^{i-1} \cdot x$. Hence, to give the map $f$ is equivalent to give just one point $x \in X_{\overline{k}}$.
Now, for every geometric point in the quotient, we want to describe its group of automorphisms. Consider a geometric point $f : G_{\overline{k}} \to X_{\overline{k}}$. An automorphism is a morphism:

$$g : G_{\overline{k}} \to G_{\overline{k}}$$

such that the diagram:

$$\begin{array}{ccc} G_{\overline{k}} & \xrightarrow{\quad g \quad} & G_{\overline{k}} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f} & \\ & X_{\overline{k}} & \end{array}$$

is commutative. Notice that giving $f$ is equivalent to give a geometric point of $X$. Since the only geometric point of $X$ invariant by the action of $G$ is the origin, there are no non-trivial automorphisms on the points corresponding to geometric points of $X$ different from the origin, and the automorphisms group of the origin is $G$ itself.

## 2.5 Coarse Moduli Spaces

Even if an algebraic stack is an object encoding more information, sometimes it can be hard to study. That's one of the reasons why we search for a *coarse moduli space*, namely an algebraic space that is close as possible to the algebraic stack.

**Definition 2.42.** Let $S$ be a scheme and $\mathcal{X}$ be an algebraic stack over $S$. A *coarse moduli space* for $\mathcal{X}$ is a couple $(X, \pi)$ where $X$ is an algebraic space and $\pi : \mathcal{X} \to X$ satisfies the following condition:

(i) If $g : \mathcal{X} \to Z$ is a morphism to an algebraic space $Z$, then there exists a unique morphism $f : X \to Z$ such that:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi} & X \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} \\ & & Z \end{array}$$

is commutative,

(ii) For every algebraically closed field $k$, the map $|\mathcal{X}(k)| \to X(k)$ is a bijection, where $|\mathcal{X}(k)|$ denotes the set of isomorphism classes in $\mathcal{X}(k)$.

This definition formalizes our previous idea of "close enough". Indeed, we require the coarse moduli space to have a map that is initial for maps to algebraic spaces, and we want it to parametrize the isomorphism classes of geometric points.

The fundamental result about the existence of the coarse moduli space of an algebraic stack is proved in [10], Keel-Mori.

**Theorem 2.43** (Keel-Mori). *Let $S$ be a locally noetherian scheme and $\mathcal{X}$ an algebraic stack locally of finite presentation over $S$ with finite diagonal. Then:*

*(a) there exists a coarse moduli space $\pi : \mathcal{X} \to X$, with $\pi$ proper,*

*(b) $X$ is locally of finite type over $S$, and if $\mathcal{X}$ is separated over $S$, then $X$ is separated over $S$,*

*(c) if $X' \to X$ is a flat morphism of algebraic spaces, then:*

$$\pi' : \mathcal{X}' = \mathcal{X} \times_X X' \to X'$$

*is a coarse moduli space for $\mathcal{X}'$.*

*Proof.* See [14] Theorem 11.1.2. □

*Remark* 2.44. In [14], Olsson remarks that Theorem 2.43 holds even without the noetherian assumption (apart from (b)). Still, from now on we are interested in algebraic stacks locally of finite presentation over locally noetherian schemes, so we don't lose anything assuming this condition by now.

It is useful to have an explicit description of the coarse moduli space, in the case of the quotient of an affine scheme by the action of a finite group.

**Proposition 2.45.** *Let $Spec(A)$ be an affine scheme and $G$ a finite group acting on it. Then:*

$$\pi : [Spec(A)/G] \to Spec\left(A^G\right)$$

*is the coarse moduli space of $[Spec(A)/G]$.*

*Proof.* By Remark 2.11 on Theorem 2.10:

$$\pi' : Spec(A) \to Spec\left(A^G\right)$$

is a categorical quotient, in the category of algebraic spaces. By Remark 2.39, the morphism:

$$q : Spec(A) \to [Spec(A)/G]$$

presents the quotient as an algebraic stack. Then, for every $g : \mathcal{X} \to Z$ morphism to an algebraic space $Z$, the composition $g \circ q$ is $G$ invariant, hence factorizes through $Spec\left(A^G\right)$. To conclude, it is easy to see that, since étale torsors over an algebraically closed field $k$ are trivial, there is a bijection on isomorphism classes of the geometric points:

$$|\,[Spec(A)/G]\,(k)| \to Spec\left(A^G\right)(k)$$

□

*Remark* 2.46. Apart from the existence of the coarse moduli space, an important point of Keel-Mori Theorem is (c); it is an example of how close the behaviour of the coarse moduli space is with respect to the starting algebraic stack. Indeed, flatness is a stable property by base change, hence it means that looking (fppf or étale) locally the algebraic stack can be done by looking (fppf or étale) locally the coarse moduli space, and then pulling it back. This fact will be crucial in the next chapter, where we are interested in the étale local behaviour of some Deligne-Mumford stacks.

### 2.5.1 Characterization of Deligne-Mumford Stacks

With the notion of coarse moduli space, we can finally give a characterization of Deligne-Mumford stacks in terms of *orbifolds*, which will be fundamental to describe twisted curves.

**Theorem 2.47.** *Let $S$ be a locally noetherian scheme and $\mathcal{X}$ a Deligne-Mumford stack locally of finite type over $S$ and with finite diagonal. Let $\pi : \mathcal{X} \to X$ be its coarse moduli space. Let $x \to \mathcal{X}$ be a geometric point, corresponding to the geometric point $\overline{x} \to X$ on the coarse moduli space. Let $G_x$ be the group of automorphisms of $x$ in $\mathcal{X}$, which is a finite group since $\mathcal{X}$ is Deligne-Mumford. Then there exists an étale neighborhood $U \to X$ of $\overline{x}$ and a finite $U$-scheme $V \to U$ with an action of $G_x$ such that:*

$$\mathcal{X} \times_X U \cong [V/G_x]$$

*Proof.* See [14] Theorem 11.3.1. □

Combining Theorem 2.47 with point (c) of Theorem 2.43, we can suppose the étale neighborhood to be affine, hence of the form $V = Spec(A)$ and $U = Spec(B)$. Moreover, we know that after the étale base change, we get that:

$$[V/G_x] \to U$$

is a presentation of the coarse moduli space. But $G_x$ is a finite group acting on an affine scheme, so by Proposition 2.45 we get that the coarse moduli space is given by the spectrum of the invariants of $A$:

$$B = A^G$$

Hence étale locally on the coarse moduli space a Deligne-Mumford stack is the quotient of an affine scheme by the action of a finite group.

# Chapter 3

# Twisted Curves

In this chapter we finally introduce the geometric objects we are interested in, *twisted curves*, and we are going to prove a characterization in terms of their geometric fibers. We characterize before the *nodal curves*, which a fortiori will be the coarse moduli spaces of twisted curves.

## 3.1 Nodal Curves

As promised, we start defining nodal curves over a base scheme $S$, and provide an equivalent characterization in terms of their geometric fibers, under the assumption that the base scheme $S$ is excellent.

**Definition 3.1.** (i) A homomorphism of rings $A \to B$ is called *regular* if it is flat and for every $p \in Spec(A)$, the fiber $B \otimes_A k(p)$ is geometrically regular over $k(p)$.

(ii) A ring $A$ is called *G-ring* if it is noetherian and for any $p \in Spec(A)$, the completion $A_p \to \widehat{A}_p$ is regular.

(iii) A ring is called *J-2 ring* if for every finitely generated $A$-algebra $B$, the set of regular points of $Spec(B)$ is open in $Spec(B)$.

(iv) A ring $A$ is called *excellent* if it is a G-ring, a J-2 ring and $Spec(A)$ is universally catenary.

(v) A scheme $S$ is called *excellent* if there exists an affine open cover $Spec(A_i)$, with $A_i$ excellent rings.

Examples of noetherian rings which are not excellent rings are difficult to construct; anyway, the excellence assumption provides numerous useful results and characterizations.

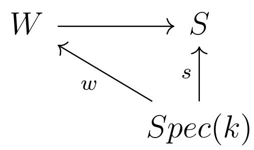
### 3.1.1 Local Picture of a Scheme

We introduce now the notion of *local picture* of a scheme at a point. This definition is essential in order to introduce the geometric object we are interested in this chapter, namely the *nodal curves*. Let's recall before the notion of *point of a scheme*, from the relative point of view we introduced in the first two chapters.

**Definition 3.2.** Let $S$ be a scheme, $k$ a field.

(i) A *$k$-point* of $S$ is a morphism $s : Spec(k) \to S$. In particular, if $k$ is algebraically closed, we will say that $s$ is a *geometric point*.

(ii) Let $W$ be a scheme and $w$ a $k$-point $w : Spec(k) \to W$. We say that the triple $(W, w, f)$ is an *étale neighborhood* of $(S, s)$, with $s : Spec(k) \to S$ a $k$-point, if $f$ is an étale morphism $f : W \to S$ such that the following diagram:




is commutative. We will often just write $(W, w)$, omitting the choice of $f$, when it is clear by the context.

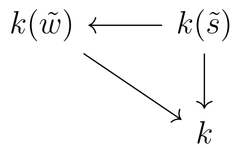
*Remark* 3.3. This definition focuses on the functorial aspect of a scheme: from this point of view, a $k$-point of a scheme is an element of:

$$S(k) = Hom_{Sch/S}(Spec(k), S)$$

Notice that to give a morphism $Spec(k) \to S$ is equivalent to give a point $\tilde{s}$ of the underlying topological space of the scheme $S$ and an extension of fields:

$$k(\tilde{s}) \to k$$

where $k(\tilde{s})$ is the residue field of $\tilde{s}$. Moreover, if $(W, w)$ is an étale neighborhood of $(S, s)$, then it means that $Spec(k)$ is mapped to a point $\tilde{w}$ of the underlying topological space of the scheme of $W$ and the morphism $W \to S$ maps $\tilde{w} \mapsto \tilde{s}$, with the following relation on the residual fields:




with $k(\tilde{s}) \to k(\tilde{w})$ an étale extension of fields.

We are mainly interested in the behaviour of geometric points. Let's recall the definition of *strict henselization* of a scheme with respect to a geometric point.

**Definition 3.4.** Let $X$ be a scheme and $\overline{x} : Spec(k) \to X$ a geometric point of $X$, corresponding to a point $x \in X$. Then the *strict henselization* of $X$ at $\overline{x}$ is:

$$X^{sh} = Spec\left(\mathcal{O}^{sh}_{X,x}\right)$$

where $\mathcal{O}^{sh}_{X,x}$ is the strict henselization of the local ring $\mathcal{O}_{X,x}$. We will use the notation $X^{sh}$ when it is clear for which geometric point we are taking the strict henselization.

*Remark* 3.5. See [15] Tag 04GP for the definition of the strict henselization of a local ring. Notice, in particular, that it makes sense to talk about the strict henselization of a geometric point $\overline{x} : Spec(k) \to X$ (corresponding to a point $x \in X$) since the definition of strict henselization requires the choice of a separable closure of the residual field $k(x)$. Indeed, working with a geometric point, we have fixed an algebraically closed extension $k(x) \to k$ of the residual field, which fixes also a separable closure of $k(x)$.

With the following lemma, we see the importance of the strict henselization of a scheme in a geometric point.

**Proposition 3.6.** *Let $S$ be a scheme and $\overline{s} : Spec(k) \to S$ be a geometric point lying over $s \in S$. Denote by $k^{sep}$ the separable closure of $k(s)$ inside $k$. Then there is a canonical identification:*

$$\mathcal{O}^{sh}_{X,x} \cong \varinjlim_{(U,\overline{u})} \mathcal{O}(U)$$

*where the direct limit is over the étale neighborhoods of $(S, \overline{s})$.*

*Proof.* See [15] Tag 04HX. □

*Remark* 3.7. This proposition shows that the strict henselization of the local ring of a point is indeed the local ring for the étale topology.

**Definition 3.8.** (i) Let $X, Y$ be schemes, $x \in X, y \in Y$. We say that the *local picture* of $X$ in $x$ is given by $Y$ in $y$ if there is a common étale neighborhood $(W, w)$ of $(X, x)$ and $(Y, y)$.

(ii) Let $f : X \to Y$ and $g : U \to V$ be morphisms of schemes, $x \in X$ and $u \in U$. We say that the *local picture* of $f : X \to Y$ in $x$ is given by $g : U \to V$ at $u$ if there is a common étale neighborhood $(W, w)$ of $(X, x)$ and $(U, u)$, and a common étale neighborhood $(Z, z)$ of $(Y, f(x))$ and $(V, g(u))$ with a map $W \to Z$ such that the following diagram:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

is commutative.

The notion of local picture of a scheme at a geometric point is deeply related to the behaviour of the strict henselization of the local ring of the point, which is indeed the local ring of the point for the étale topology.

**Lemma 3.9.** *Let $X, Y$ be schemes and suppose that the local picture of $X$ in $\overline{x} \to X$ is given by $Y$ in $\overline{y} \to Y$, with $\overline{x}, \overline{y}$ geometric points. Then $X^{sh} \cong Y^{sh}$, where the strict henselization is taken with respect to the points $\overline{x}, \overline{y}$.*

*Proof.* Let $(W, w)$ be a common étale neighborhood of $(X, \overline{x})$ and $(Y, \overline{y})$. Since the strict henselization in a point is the local ring for the étale topology, hence a direct limit of étale neighborhoods, it means that $\mathcal{O}^{sh}_{X,x} \cong \mathcal{O}^{sh}_{Y,y}$, so $X^{sh} \cong Y^{sh}$. □

**Lemma 3.10.** *Let $S$ be a scheme and $X, Y$ be schemes locally of finite presentation over $S$, $\overline{x} \to X, \overline{y} \to Y$ be geometric points. If $X^{sh} \cong Y^{sh}$ (with respect to $\overline{x}$ and $\overline{y}$), then the local picture of $X$ in $\overline{x}$ is given by $Y$ in $\overline{y}$.*

*Proof.* The geometric points $\overline{x}, \overline{y}$ correspond to points $x \in X, y \in Y$. Suppose that locally the schemes $X$ over $S$ around $x$ is given by $Spec(A) \to Spec(R)$, and $Y$ around $y$ by $Spec(B) \to Spec(R)$. Then we have a presentation:

$$A = \frac{R[x_1, ..., x_n]}{(f_1, ..., f_r)}$$

Consider:

$$A \to \mathcal{O}^{sh}_{X,x} \to \mathcal{O}^{sh}_{Y,y} = \varinjlim B_i = \bigsqcup_{B \to B_i} B_i/\sim$$

where $B \to B_i$ are étale $B$-algebras, and $\sim$ is the usual equivalence relation presenting the direct limit. Let $z_i$ be the images of $x_i$ in $\bigsqcup_{B \to B_i} B_i/\sim$ for $i = 1, ..., n$. Then each $z_i \in B_i$, for suitable $B_i$; setting:

$$B' = B_1 \times_B .. \times_B B_n$$

we get a morphism $A \to B'$. In the same way, we construct a morphism $B \to A'$. Since $\mathcal{O}^{sh}_{X,x} \cong \mathcal{O}^{sh}_{Y,y}$, we can find a common étale algebra $C$ of $A', B'$. Hence, $Spec(C)$ is the common étale neighborhood of $Spec(A), Spec(B)$. □

**Corollary 3.11.** *Let $X, Y$ be schemes locally of finite presentation over a scheme $S$, $\overline{x} \to X, \overline{y} \to Y$ be geometric points. Then the following are equivalent:*

*(a) The local picture of $X$ at $\overline{x}$ is given by $Y$ at $\overline{y}$,*

*(b) $X^{sh} \cong Y^{sh}$.*

*Proof.* Combine Lemma 3.9 and Lemma 3.10. □

### 3.1.2 Nodal Curves

We can finally define the notion of nodal curve over a base scheme $S$.

**Definition 3.12.** Let $S$ be a scheme. A *nodal curve* over $S$ is a morphism $f : C \to S$ locally of finite presentation, such that for every geometric point $c$ of $C$, the local picture of $C \to S$ at $c$ is given by $Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right) \to Spec(A)$ at a suitable point, with $A$ a ring and $\alpha \in A$.

*Remark* 3.13. Let $k$ be a field. Liu in [12] defines a *semistable curve* over $k$ a curve $C$ over $k$ such that the extension $C_{\overline{k}}$ to the algebraic closure $\overline{k}$ is reduced and its singular points are ordinary double points. Notice that by Liu, [12] Proposition 7.5.15, for a point $c \in C$, it is equivalent to be an ordinary double point and the following condition on the local ring:

$$\widehat{\mathcal{O}}_{C_{\overline{k}},c} \cong \frac{\overline{k}[\![x,y]\!]}{(xy)}$$

which is indeed equivalent to the condition given in the definition of nodal curve given above.

**Example 3.14.** Let $k$ be a field with $\mathrm{char}(k) \neq 2$. Let $p(x) \in k[x]$ be a polynomial such that all its roots in $\overline{k}$ are of order at most 2, and consider the affine plane curve:

$$U = Spec\left(\frac{k[x,y]}{(y^2 - p(x))}\right)$$

Then $U$ is nodal (namely semistable, in Liu's definition). In fact, consider the extension $U_{\overline{k}} = Spec\left(\frac{\overline{k}[x,y]}{(y^2-p(x))}\right)$. Let $u = (a,b)$ be a point of this curve; by the jacobian criterion, $(a,b)$ is singular if and only if $p(a) = 0$ and $p'(a) = 0$. This means that $y^2 = (x-a)^2 q(x)$, with $q(a) \neq 0$, for $q(x) \in \overline{k}[x]$. Since $q(a) \neq 0$, we can find $h(x) \in \overline{k}[\![x-a]\!]^{\times}$ such that $q(x) = h(x)^2$. Hence we deduce that:

$$\widehat{\mathcal{O}}_{C_{\overline{k}},u} = \frac{\overline{k}[\![x-a,y]\!]}{(y^2 - p(x))} = \frac{\overline{k}[\![x-a,y]\!]}{(y-(x-a)h(x))(y+(x-a)h(x))} \cong \frac{\overline{k}[\![v,w]\!]}{(vw)}$$

It is important to notice the different behaviour of the completion of the local ring at an ordinary double point of a nodal curve over a non algebraically closed field, as the next example is showing.

**Example 3.15.** Let $p \neq 2$ be a prime such that $p \not\equiv 1 \bmod (4)$, and consider the curve $C$ over $\mathbb{F}_p$ of equation $x^2 + y^2 = 0$. Setting $c$ the origin, we have that:

$$\widehat{\mathcal{O}}_{C,c} = \frac{\mathbb{F}_p[\![x,y]\!]}{(x^2+y^2)} \not\cong \frac{\mathbb{F}_p[\![v,w]\!]}{(vw)}$$

However, if we extend the base field $\mathbb{F}_p \hookrightarrow \mathbb{F}_p[i] = \mathbb{F}_{p^2}$, where $i$ is a solution of $T^2 + 1 = 0$, then we get:

$$\widehat{\mathcal{O}}_{C_{\mathbb{F}_{p^2}},c} = \frac{\mathbb{F}_{p^2}[\![x,y]\!]}{(x+iy)(x-iy)} \cong \frac{\mathbb{F}_{p^2}[\![v,w]\!]}{(vw)}$$

Hence the curve $C$ is a nodal curve, but sometimes the ordinary double points cannot be seen if we don't extend the field to its algebraic closure. However, the following results tell us that we are able to find these ordinary double points just base changing to a finite separable extension of the base field.

In view of this alternative definition of nodal curve, recall this proposition, which gives a characterization of singular points of a nodal curve over a field.

**Proposition 3.16.** *Let $C$ be a nodal curve over a field $k$, $c \in C$ be a singular point and $\pi : C' \to C$ be the normalization morphism.*

*(a) For $c' \in \pi^{-1}(c)$, $k(c), k(c')$ are separable extensions of $k$.*

*(b) Suppose the points in $\pi^{-1}(c)$ are rational over $k$. Then:*

$$\widehat{\mathcal{O}}_{C,c} \cong \frac{k[\![x, y]\!]}{(xy)}$$

*Proof.* See Liu [12], Proposition 10.3.7. □

**Corollary 3.17.** *Let $C$ be a nodal curve over a field $k$, $c \in C$ be a singular point. Then there exists a finite separable extension $k \to L$ such that:*

$$\widehat{\mathcal{O}}_{C_L,c} \cong \frac{L[\![x, y]\!]}{(xy)}$$

*Proof.* By Proposition 3.16 (a), the fields $k(c), k(c')$ are separable and finite, for all $c' \in \pi^{-1}(c)$, with $\pi$ the normalization morphism. Set $L$ the smallest field containing $k(c)$ and $k(c')$ for any $c' \in \pi^{-1}(c)$. Now consider the curve $C_L$ over $L$. By construction, the points $c' \in \pi^{-1}(c)$ are rational over $L$, hence by Proposition 3.16 (b), we have that $\widehat{\mathcal{O}}_{C_L,c} \cong \frac{L[\![x,y]\!]}{(xy)}$. □

This means that we can detect possible double ordinary points looking into a finite separable, namely étale, extension of the base field $k$, without passing to its algebraic closure.

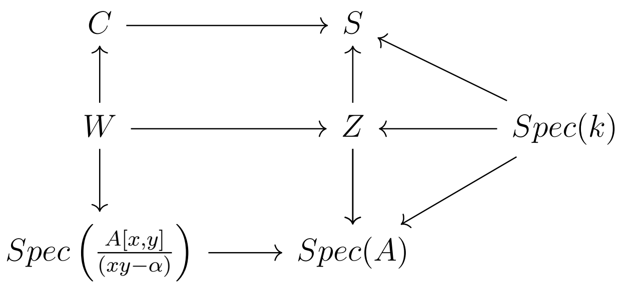
## 3.2 Characterization of Nodal Curves

With the following Propositions, we are going to prove Corollary 3.26, which gives an equivalent characterization of a nodal curve $C \to S$ in terms of its geometric fibers.

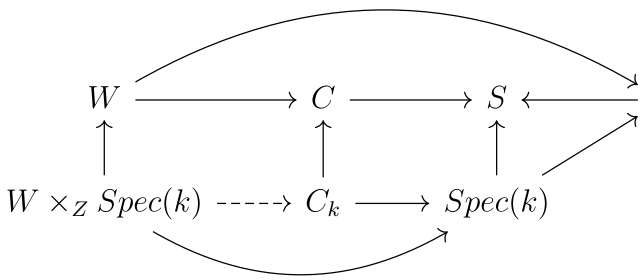
**Proposition 3.18.** *Let $S$ be a scheme, $C \to S$ be a morphism of schemes. If $C \to S$ is a nodal curve, then all its geometric fibers are nodal curves.*

*Proof.* Let $Spec(k) \to S$ be a geometric point of $S$, $C_k = C \times_S Spec(k)$ be the corresponding fiber; we want to prove that $C_k \to Spec(k)$ is a nodal curve. Let $c$ in $C_k$ be a geometric point in the fiber (and hence of $C$). By hypothesis, the local picture of $C \to S$ at $c$ is given by $Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right) \to Spec(A)$ at a suitable point, with $\alpha \in A$, hence there are étale neighborhoods $(W, w), (Z, z)$ and a morphism $W \to Z$

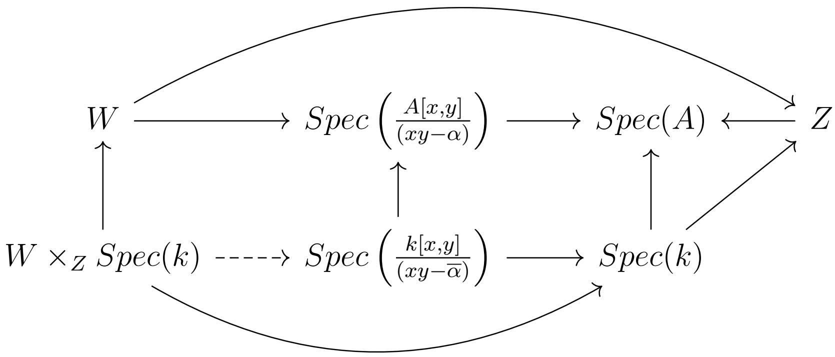
as in the definition such that:




is commutative. Consider the diagram:




where the dotted arrow exists by universal property, and similarly:




We conclude deducing that $W \times_Z Spec(k)$ is the common étale neighborhood such that the local picture of $C_k \to Spec(k)$ is given by $Spec\left(\frac{k[x,y]}{(xy-\overline{\alpha})}\right) \to Spec(k)$.

□

The hardest part is to prove the converse, namely, starting from the information on the fibers, which are nodal curves, to get a local picture of the whole family of nodal curves. In the following lemma, we describe the completion of the local ring of a node, in order to apply later Lemma 3.23. The proof will follow the ideas of Liu in [12] Lemma 10.3.20.

**Lemma 3.19.** *Let $\rho : A \to B$ be a flat local homomorphism of local noetherian rings, which induces an isomorphism $k = A/m_A \cong B/m_B$ and such that $\widehat{(B/m_A B)} \cong \frac{k[\![x,y]\!]}{(xy)}$. Then there exists $\alpha \in m_A$ such that $\widehat{B} \cong \frac{\widehat{A}[\![x,y]\!]}{(xy-\alpha)}$.*

*Proof.* For simplicity, we'll denote by $a$ the image $\rho(a)$ of an element $a \in A$. First of all, notice that by hypothesis:

$$(B/m_A B)^{\widehat{}} \cong \frac{k[\![x,y]\!]}{(xy)}$$

Call $u, v$ the images via the isomorphism of $x, y$. But since $x, y$ generate the maximal ideal of $\frac{k[\![x,y]\!]}{(xy)}$, then $u, v$ generate the maximal ideal of $(B/m_A B)^{\widehat{}}$ (and consequently the one of $B/m_A B$), hence we get that $m_B = uB + vB + m_A B$. Moreover, notice that since $xy = 0$ in $\frac{k[\![x,y]\!]}{(xy)}$, then $uv \in m_A B$.
Our first aim is to construct a sequence of elements $(u_n)_n, (v_n)_n, (\epsilon_n)_n$ with $u_n, v_n \in m_B$, $\epsilon_n \in m_A$ such that:

- $u_0 = u, v_0 = v$,
- $u_{n+1} - u_n \in m_A^{n+1} B, v_{n+1} - v_n \in m_A^{n+1} B, \epsilon_{n+1} - \epsilon_n \in m_A^{n+1}$,
- $u_n v_n - \rho(\epsilon_n) \in m_A^{n+1} B$.

Let's work by induction: take $u_0 = u, v_0 = v, \epsilon_0 = 0$ and suppose to have constructed $u_n, v_n, \epsilon_n$. We have that $uB + vB \subseteq u_n B + v_n B + m_A B$ hence:

$$\begin{aligned} m_A^{n+1} B &\subseteq \rho(m_A^{n+1}) + m_A^{n+1} B m_B = \rho(m_A^{n+1}) + m_A^{n+1} B (uB + B + m_A B) \subseteq \\ &\subseteq \rho(m_A^{n+1}) + m_A^{n+1} B (u_n B + v_n B + m_A B) = \\ &= \rho(m_A^{n+1}) + u_n m_A^{n+1} B + v_n m_A^{n+1} B + m_A^{n+2} B \end{aligned}$$

In this decomposition, we can write $(u_n v_n - \epsilon_n) = \delta_{n+1} + u_n b_n + v_n c_n + d_n$. Then:

$$(u_n - c_n)(v_n - b_n) = (\epsilon_n + \delta_{n+1}) + (b_n c_n + d_n)$$

and so $u_{n+1} = u_n - c_n, v_{n+1} = v_n - b_n, \epsilon_{n+1} = \epsilon_n + \delta_{n+1}$ satisfy the required properties.
For every $n \geq 0$, consider the homomorphism:

$$\phi_n : \frac{A[x,y]}{(m_A, x, y)^n} \to B/m_B^n$$

that sends $x, y$ to $u_n, v_n$, and denote by $I_n = (m_A, x, y)^n$. By the properties of $u_n, v_n$, the homomorphisms are well defined and they induce a homomorphism of projective systems:

$$(A[x,y]/I_n)_n \to (B/m_B^n)_n$$

As:

$$m_B = m_A B + u_n B + v_n B = m_A + u_n A + v_n A + m_B^2$$

we deduce that the $\phi_n$ are surjective. In fact, fix $n \in \mathbb{N}$ and take $b \in B/m_B^n$; a representative is $b = b_0 + ... + b_{n-1}$, with $b_i \in m_B^i$. But by the above relation, every element in $m_B$ is image of an element of $\frac{A[x,y]}{(m_A,x,y)^n}$ plus an element of $m_B^2$. Hence:

$$b = \phi_n(a) + \bar{b}_2 + ... + b_{n-1}$$

with $\bar{b}_2 \in m_B^2$. Iterating this reasoning, using that:

$$m_B^i = m_B^{i-1}(m_A + u_n A + v_n A) + m_B^{i+1}$$

we get that $b$ is in the image of $\phi_n$. This implies that $\phi_{n+1}(I_n) = m_B^n/m_B^{n+1}$, hence the exact sequence of projective systems:

$$0 \to (\ker \phi_n)_n \to (A[x,y]/I_n)_n \to (B/m_B^n)_n \to 0$$

induces the exact sequence on the projective limits (by [12] Liu, Lemma 1.3.1):

$$0 \to \varprojlim(\ker \phi_n) \to \varprojlim\,(A[x,y]/I_n) \to \varprojlim\,(B/m_B^n) \to 0$$

Hence we have an induced surjection $\phi : \widehat{A}[\![x,y]\!] \to \widehat{B}$. Let $\alpha \in \widehat{A}$ the element induced by the sequence $(\epsilon_n)_n$. In particular, $\alpha \in \varprojlim\,(m_A/m_A^n) \cong m_A\widehat{A}$. By construction, $\phi(x)\phi(y) = \phi(\alpha)$, hence $\phi$ induces a surjection $\bar{\phi} : \frac{\widehat{A}[\![x,y]\!]}{(xy-\alpha)} \to \widehat{B}$. We want to reduce now to $\alpha \in m_A$. For a certain $\delta \in m_A\widehat{A}$, we have $(1+\delta)\alpha \in m_A$, hence:

$$\frac{\widehat{A}[\![x,y]\!]}{(xy-\alpha)} = \frac{\widehat{A}[\![x,(1+\delta)y]\!]}{(x(1+\delta)y-(1+\delta)\alpha)} \cong \frac{\widehat{A}[\![x,y]\!]}{(xy-(1+\delta)\alpha)}$$

so we can assume $\alpha \in m_A$. Finally, we prove the injectiveness of $\bar{\phi}$. We tensor the exact sequence:

$$0 \to \ker \bar{\phi} \to \frac{\widehat{A}[\![x,y]\!]}{(xy-\alpha)} \to \widehat{B} \to 0$$

by $\widehat{A}/m_A\widehat{A}$, and since $\widehat{A} \to \widehat{B}$ is flat, then the following sequence is exact (by [12] Liu, Proposition 1.2.6):

$$0 \to \ker \bar{\phi}/m_A \ker \bar{\phi} \to \frac{k[\![x,y]\!]}{(xy)} \to (B/m_A B)^{\widehat{}} \to 0$$

The right arrow is an isomorphism by hypothesis, so $\ker \bar{\phi} = m_A \ker \bar{\phi}$, which implies that $\ker \bar{\phi} = 0$ by Nakayama Lemma. □

Recall the following two Lemmas. They provide the existence of a common étale neighborhood of two schemes, under suitable conditions.

**Lemma 3.20.** *Let $\phi : X \to Y$ be a morphism of schemes. Let $x \in X$ and let $V \subset Y$ be an affine open neighbourhood of $\phi(x)$. If $\phi$ is smooth at $x$, then there exists an integer $d \geq 0$ and an affine open $U \subset X$ with $x \in U$ and $\phi(U) \subset V$ such that there exists a commutative diagram:*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{\pi} & \mathbb{A}^d_V \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & & \end{array}$$

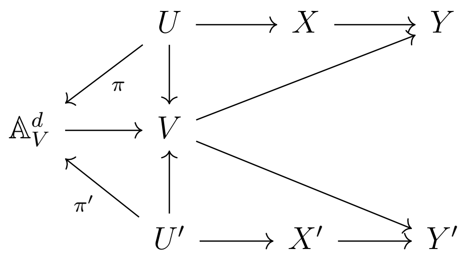
*where $\pi$ is étale.*

*Proof.* See [15] Stacks Project, Tag 054L. □

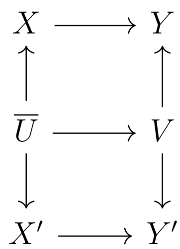
*Remark* 3.21. The integer $d$ is effectively computable, as the rank of the Jacobian in the point $x$. In other words, smooth schemes are étale locally like affine spaces of the expected dimension.

**Corollary 3.22.** *Let $f : X \to Y$, $g : X' \to Y'$ be morphisms of schemes, smooth and of relative dimension $d$ in $x \in X, x' \in X'$ such that $Y, Y'$ have a common open neighborhood $V$. Then the local picture of $X \to Y$ at $x$ is given by $X' \to Y'$ at $x'$.*

*Proof.* By Lemma 3.20 and Remark 3.21, there exist an affine open $U \subseteq X$ of $x$ and an affine open $U' \subseteq X'$ of $x'$, with maps $\pi, \pi'$ as in the Lemma, such that the diagram:




is commutative. Since étale neighborhoods of $V$ form a direct set, there exists a common étale neighborhood $\overline{U}$ of $U, U'$. Hence by the commutative diagram:




the local picture of $X \to Y$ at $x$ is given by $X' \to Y'$ at $x'$. □

The second Lemma relies heavily on the assumption of excellence of the schemes and shows how, given $x \in X, y \in Y$ and an isomorphism on the completions of two local rings, we can find a common étale neighborhood of the schemes.

**Lemma 3.23.** *Let $S$ be a locally noetherian scheme and $X, Y$ schemes locally of finite type over $S$. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume we have an $\mathcal{O}_{S,s}$-algebra isomorphism:*

$$\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$$

*between the completion of the local rings. Then for every $N \geq 1$ there exists a common étale neighborhood $(U, u)$ of $(X, x)$ and $(Y, y)$ such that the diagram:*

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{U,u} & & \\ \uparrow & \nwarrow & \\ \widehat{\mathcal{O}}_{Y,y} & \longrightarrow & \widehat{\mathcal{O}}_{X,x} \end{array}$$

*commutes modulo $m_u^N$.*

*Proof.* See [15] Stacks Project, Tag 0CAV. □

*Remark* 3.24. If $S$ is an excellent scheme, it is in particular locally noetherian and the local rings are G-rings, hence it satisfies the more general hypothesis of this Lemma.

**Theorem 3.25.** *Let $S$ be an excellent scheme and $C \to S$ be a flat morphism locally of finite presentation, such that all the geometric fibers are nodal curves. Then $C \to S$ is a nodal curve.*

*Proof.* Suppose that locally around a geometric point $c$ of $C$ the morphism is given by $f : Spec(B) \to Spec(A)$. The point $c$ corresponds to an ideal $m_B$ of $B$ and $f(c)$ to an ideal $m_A$ of $A$. Thus, the map on the rings $A \to B$ induces a map on the localizations $A_{m_A} \to B_{m_B}$. Set $k = A_{m_A}/m_{A_{m_A}} \cong B_{m_B}/m_{B_{m_B}}$; by hypothesis, the local picture of the geometric fiber $Spec(B \otimes_A \overline{k}) \to Spec(\overline{k})$ is given by $Spec\left(\frac{\overline{k}[x,y]}{(xy-a)}\right) \to Spec(\overline{k})$ at a suitable point, for $a \in \overline{k}$.
If $a \neq 0$, then $Spec\left(\frac{\overline{k}[x,y]}{(xy-a)}\right) \to Spec(\overline{k})$ is smooth. Moreover, the map induced on the local rings $A_{m_A} \to B_{m_B}$ is flat, hence for [15] Tag 01V9, $C \to S$ is smooth in the point $c$; since it is of relative dimension 1, by Corollary 3.22, the local picture of $C \to S$ in $c$ is given by $Spec\left(\frac{A[x,y]}{(xy-1)}\right) \to Spec(A)$ at a suitable point.
If $a = 0$, then $Spec\left(\frac{\overline{k}[x,y]}{(xy)}\right) \to Spec(\overline{k})$ has a singular point in the origin. By hypothesis on the local picture of the geometric fiber we get:

$$\left(B_{m_B} \otimes_{A_{m_A}} \overline{k}\right)^{\widehat{}} \cong \frac{\overline{k}[\![x, y]\!]}{(xy)}$$

By Corollary 3.17 we find a suitable étale field extension $k \to L$, namely a finite separable extension $L$ of $k$, such that the following holds:

$$\left(B_{m_B} \otimes_{A_{m_A}} L\right)^{\widehat{}} \cong \frac{L[\![x, y]\!]}{(xy)}$$

Consider the base change $Spec(A \otimes_k L) \to Spec(A)$, which is étale since the extension $k \to L$ is étale and the base change $Spec(B \otimes_k L) \to Spec(B)$, which is étale since $Spec(A \otimes_k L) \to Spec(A)$ is étale. Define $\tilde{B} = B \otimes_k L$ and $\tilde{A} = A \otimes_k L$. Now, the morphism $Spec(\tilde{B}) \to Spec(\tilde{A})$ is flat, since flatness is stable by base change, and induces a flat local homomorphism on the local rings $\tilde{A}_{m_{\tilde{A}}} \to \tilde{B}_{m_{\tilde{B}}}$, which satisfy the hypothesis of Lemma 3.19 since:

$$\left(\tilde{B}_{m_{\tilde{B}}} \otimes_{\tilde{A}_{m_{\tilde{A}}}} L\right)^{\widehat{}} = \left(B_{m_B} \otimes_k L \otimes_{A_{m_A} \otimes_k L} L\right)^{\widehat{}} \cong \left(B_{m_B} \otimes_{A_{m_A}} L\right)^{\widehat{}} \cong \frac{L[\![x,y]\!]}{(xy)}$$

hence we conclude that:

$$\widehat{\tilde{B}_{m_{\tilde{B}}}} \cong \frac{\widehat{\tilde{A}_{m_{\tilde{A}}}}[\![x,y]\!]}{(xy-\alpha)}$$

with $\alpha \in \tilde{A}_{m_{\tilde{A}}}$. Moreover, since $m_{\tilde{A}}$ is a maximal ideal in $\tilde{A}_{m_{\tilde{A}}}$, by multiplying $\alpha$ for a suitable invertible element we may suppose $\alpha \in \tilde{A}$. This means that $Spec(\tilde{B})$ and $Spec\left(\frac{\tilde{A}[x,y]}{(xy-\alpha)}\right)$ have isomorphic completions of the local rings:

$$\widehat{\mathcal{O}}_{Spec(\tilde{B}),m_{\tilde{B}}} \cong \widehat{\mathcal{O}}_{Spec\left(\frac{\tilde{A}[x,y]}{(xy-\alpha)}\right),(m_{\tilde{A}},x,y)}$$

Notice that A is an excellent ring and $\tilde{A}$, $\tilde{B}$, $\frac{\tilde{A}[x,y]}{(xy-\alpha)}$ are $A$-algebras of finite type, hence they are excellent and satisfy the hypotheses of Lemma 3.23, so we can find a common étale neighborhood $(Y,y)$ of $(Spec(\tilde{B}), m_{\tilde{B}})$ and $\left(Spec\left(\frac{\tilde{A}[x,y]}{(xy-\alpha)}\right), (m_{\tilde{A}}, x, y)\right)$. It follows that the diagram:

$$\begin{array}{ccccccc}
Spec(\tilde{B}) & \longrightarrow & Spec(B) & \longrightarrow & C & \longrightarrow & S \\
\uparrow & & & & & & \uparrow \\
Y & & & \longrightarrow & & & Spec(\tilde{A}) \\
\downarrow & & & & & \nearrow & \\
Spec\left(\frac{\tilde{A}[x,y]}{(xy-\alpha)}\right) & & & & & &
\end{array}$$

is commutative and it simplifies to:

$$\begin{array}{ccc}
C & \longrightarrow & S \\
\uparrow & & \uparrow \\
Y & \longrightarrow & Spec(\tilde{A}) \\
\downarrow & & \downarrow \\
Spec\left(\frac{\tilde{A}[x,y]}{(xy-\alpha)}\right) & \longrightarrow & Spec(\tilde{A})
\end{array}$$

We conclude by noticing that by construction, $Y$ and $Spec(\tilde{A})$ are the étale neighborhoods required to make $C \to S$ a nodal curve. $\square$

**Corollary 3.26.** *Let $S$ be an excellent scheme and $C \to S$ be a flat morphism locally of finite presentation. Then the following are equivalent:*

*(a) $C \to S$ is a nodal curve,*

*(b) all the geometric fibers of $C \to S$ are nodal curves.*

*Proof.* Combine Proposition 3.18 and Theorem 3.25. □

# 3.3 Twisted Curves

## 3.3.1 Local Picture of an Algebraic Stack

We generalize the definition of local picture of a scheme given in the beginning of the chapter to algebraic stacks.

**Definition 3.27.** Let $\mathcal{X}$ be an algebraic stack admitting a coarse moduli space $X$. Let $x \to \mathcal{X}$ be a geometric point with image $\overline{x} \to X$. We denote by:

$$\mathcal{X}^{sh} = \mathcal{X} \times_X X^{sh}$$

the *strict henselization* of the algebraic stack $\mathcal{X}$, where $X^{sh}$ is the strict henselization with respect to the point $\overline{x}$.

**Definition 3.28.** (i) Let $\mathcal{X}, \mathcal{Y}$ be algebraic stacks admitting coarse moduli spaces $X, Y$ and let $x \to \mathcal{X}, y \to \mathcal{Y}$ be geometric points with image $\overline{x} \to X, \overline{y} \to Y$. We say that the *local picture* of $\mathcal{X}$ at $x$ is given by $\mathcal{Y}$ at $y$ if:

$$\mathcal{X}^{sh} \cong \mathcal{Y}^{sh}$$

(ii) Let $f : \mathcal{X} \to \mathcal{Y}, g : \mathcal{U} \to \mathcal{V}$ be a morphism of algebraic stacks admitting coarse moduli spaces $X, Y, U, V$ and $x \to \mathcal{X}, u \to \mathcal{U}$ be geometric points. We say that the *local picture* of $f$ at $x$ is given by $g$ at $u$ if there are isomorphisms $\mathcal{X}^{sh} \cong \mathcal{U}^{sh}$, $\mathcal{Y}^{sh} \cong \mathcal{V}^{sh}$ respectively in the points $x, u, f(x), g(u)$ such that the following diagram:

$$\begin{array}{ccc} \mathcal{X}^{sh} & \longrightarrow & \mathcal{Y}^{sh} \\ \downarrow & & \downarrow \\ \mathcal{U}^{sh} & \longrightarrow & \mathcal{V}^{sh} \end{array}$$

is commutative.

*Remark* 3.29. Let's see the connection with the notion of local picture for schemes we introduced in the beginning of the chapter. If $\overline{x} \to X$ is a geometric point of a scheme, $X$ is naturally an algebraic stacks, and $id : X \to X$ is its coarse moduli space. Hence $X \times_X X^{sh} \cong X^{sh}$ (with respect to $\overline{x}$), so the definition of strict henselization of an algebraic stack coincides with the one for schemes. Hence, asking that two schemes $X, Y$ have the same local picture at a geometric point is to ask that $X^{sh} \cong Y^{sh}$, with respect to the geometric points taken into consideration. Notice that in the last sections we preferred to avoid the use of strict henselizations and work with common étale neighborhoods; but by Corollary 3.11, under the assumption of locally of finite presentation over the base scheme, the two notions are equivalent.

### 3.3.2 Twisted Curves

In Example 2.7, we studied the action of the group of $r$-roots of unity on $Spec\left(\frac{k[x,y]}{(xy)}\right)$, with $k$ an algebraically closed field whose characteristic does not divide $r$. Take now a ring $A$ containing the $r$-roots of unity and fix one of them $\xi$. Let $\alpha \in A$ and set $X = Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)$, which is a scheme over $S = Spec(A)$ Suppose moreover that $r$ is invertible in $A$. Let $\mu_r = \bigsqcup_{i=1}^{r} Spec(A)$ be a finite discrete group as in Example 2.7. We endow $X$ of an action of $\mu_r$:

$$\mu_r \times X \to X \tag{3.1}$$

is equivalent to:

$$\bigsqcup_{i=1}^{r} X \to X$$

so is the datum of maps $\alpha_i : X \to X$, $i = 1, ..., r$. $X$ is affine, so we can describe the maps on the global sections:

$$\alpha_i : \frac{A[x,y]}{(xy-\alpha)} \to \frac{A[x,y]}{(xy-\alpha)}$$

given by $x \mapsto \xi^i x$ and $y \mapsto \xi^{-i} y$. Following a similar construction in Example 2.41, let $k$ be an algebraic closed field. Then the isomorphism classes of $k$-points of $\left[Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)/\mu_r\right]$ are in bijection with the points of $Spec\left(\frac{A[x,y]}{(xy-\alpha^r)}\right)$. Moreover, such a geometric point has a non-trivial automorphism group if and only if it corresponds to a point whose corresponding ideal contains $x, y$; in the easier case of Example 2.41, the only point with non-trivial automorphism group is in fact the origin, corresponding to the ideal $(x, y)$.

**Definition 3.30.** A stack $\mathcal{X}$ is called *tame* if for every geometric point $x : Spec(k) \to \mathcal{X}$, $char(k)$ does not divide the order of the group of automorphisms of $x$.

**Definition 3.31.** Let $S$ be a scheme and $\mathcal{C}$ a tame algebraic stack over $S$ locally of finite presentation and with finite diagonal, with coarse moduli space $C$. We say that $\mathcal{C}$ is a *twisted curve* if for every geometric point $c \to \mathcal{C}$, the local picture at $c$ is given by:

$$\left[Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)/\mu_r\right] \to Spec(A)$$

at a suitable point, with $A$ a ring and $\alpha \in A$, where the action of $\mu_r$ is the one described in (3.1).

*Remark* 3.32. By the given étale local characterization of a twisted curve, we get that a twisted curve is in fact a Deligne-Mumford stack (see [14] Theorem 8.3.3).

We are going to describe now the principal features of a twisted curve. The assumption on the base scheme $S$ and the morphism $\mathcal{C} \to S$ are necessary to use Keel-Mori Theorem 2.43.

**Proposition 3.33.** *Let $S$ be a locally noetherian scheme and $\mathcal{C}$ a twisted curve over $S$. Then its coarse moduli space $C$ is a nodal curve over $S$.*

*Proof.* By definition, the local picture of $\mathcal{C}$ at a geometric point $c \to \mathcal{C}$ is given by:

$$\mathcal{X} = \left[ Spec \left( \frac{A[x,y]}{(xy-\alpha)} \right) / \mu_r \right] \to Spec(A)$$

with $\alpha \in A$, at a suitable geometric point; this means that $\mathcal{C}^{sh} \cong \mathcal{X}^{sh}$ (with respect to these geometric points). Let $C$ be the coarse moduli space of $\mathcal{C}$ and $X$ the coarse moduli space of $\mathcal{X}$. Since the map from a local ring to its strict henselization is flat, also the morphism $X \to X^{sh}$ from a scheme to its strict henselization with respect to a geometric point is flat. Hence applying Theorem 2.43 (c), we get that $X^{sh}$ is the coarse moduli space of $\mathcal{X}^{sh}$ and $C^{sh}$ is the coarse moduli space of $\mathcal{C}^{sh}$. But since $\mathcal{C}^{sh} \cong \mathcal{X}^{sh}$, also their coarse moduli spaces are isomorphic. We conclude noticing that by Corollary 3.11 it is equivalent to say that the local picture of $C \to S$ is given by $Spec \left( \frac{A[x,y]}{(xy-\alpha)} \right) \to Spec(A)$, making so $C \to S$ a nodal curve. □

**Proposition 3.34.** *Let $S$ be a locally noetherian scheme and $\mathcal{C}$ a twisted curve over $S$, with coarse moduli space $C$. Then for every geometric point $c \to \mathcal{C}$, there exists an étale neighborhood $W$ of the induced geometric point $\overline{c} \to C$ such that:*

$$\mathcal{C} \times_C W \cong [V/\mu_r]$$

*with $V$ a nodal curve with the action of the group of roots of unity $\mu_r$.*

*Proof.* By Theorem 2.43 and Theorem 2.47, without loss of generality we can suppose $\mathcal{C} \cong [V/\mu_r]$, by étale base changing the coarse moduli space, with $V$ a scheme. Moreover, since $\mathcal{C}$ is a twisted curve, its local picture at a geometric point is given by:

$$\mathcal{X} = \left[ U/\mu_r \right] \to Spec(A)$$

at a suitable geometric point, with $U = Spec \left( \frac{A[x,y]}{(xy-\alpha)} \right)$, $\alpha \in A$, namely $\mathcal{C}^{sh} \cong \mathcal{X}^{sh}$, which means:

$$\left[ V \times_C C^{sh} / \mu_r \right] \cong \left[ U \times_X X^{sh} / \mu_r \right]$$

where $X = Spec \left( \left( \frac{A[x,y]}{(xy-\alpha)} \right)^{\mu_r} \right)$ is the coarse moduli space of $U$. The two quotients are presented as Deligne-Mumford stacks by étale surjections:

$$\begin{array}{ccc} V \times_C C^{sh} & \twoheadrightarrow & \left[ V \times_C C^{sh} / \mu_r \right] \\ & & \downarrow \wr \\ U \times_X X^{sh} & \twoheadrightarrow & \left[ U \times_X X^{sh} / \mu_r \right] \end{array}$$

Since both $V \times_C C^{sh}$ and $U \times_X X^{sh}$ have an étale morphism to a common object through the isomorphism, then:

$$\left(V \times_C C^{sh}\right)^{sh} \cong \left(U \times_X X^{sh}\right)^{sh}$$

which yields:

$$V^{sh} \cong U^{sh}$$

making $V$ a nodal curve. □

Conversely, as we can expect, we can prove that the quotient of a nodal curve by a group of roots of unity is a twisted curve, under the hypothesis of tameness. In particular, given a nodal curve $V \to S$, whose local picture at a geometric point is given by $Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right) \to Spec(A)$ at a suitable geometric point, we say that an action of a group $\mu_r$ on $V$ is *compatible with the local picture* if there exists an isomorphism:

$$V^{sh} \cong Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)^{sh}$$

$\mu_r$-equivariant, with the action of $\mu_r$ on $Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)$ as in (3.1) and the characteristic of the residue field of any point does not divide $r$. Let's prove a lemma before this result.

**Lemma 3.35.** *Let $R$ be a ring with the action of a finite group $G$, $m$ a prime ideal of $R$ and fix $k^{sep}$ a separable closure of the residue field of $R_m$. Denote by $\left(R^G\right)^{sh}$ the strict henselization of $R^G_m$, with respect to the fixed separable closure $k^{sep}$. Then:*

$$R \otimes_{R^G} \left(R^G\right)^{sh} \cong \varinjlim_{R \to R_i} R_i$$

*where the direct limit is over the filtered category of pairs $(R_i, q)$, with $R_i$ a $G$-equivariant étale $R$-algebra and $q$ an ideal lying over $m$.*

*Proof.* We know that:

$$\left(R^G\right)^{sh} \cong \varinjlim_{R^G \to R^G_i} R^G_i$$

where the limit is over the filtered category of pairs $(R^G_i, q)$, with $R^G_i$ an étale $R^G$-algebra and $q$ an ideal lying over $m$ (morphisms between two pairs are morphisms of algebras respecting the ideals, see [15] Tag 04GP for a precise description of this direct limit). Since direct limits commute with tensor product:

$$R \otimes_{R^G} \varinjlim_{R^G \to R^G_i} R^G_i \cong \varinjlim_{R^G \to R^G_i} \left(R \otimes_{R^G} R^G_i\right)$$

To conclude, we need to show that these two limits are isomorphic:

$$\varinjlim_{R^G \to R^G_i} \left(R \otimes_{R^G} R^G_i\right) \cong \varinjlim_{R \to R_i} R_i$$

over the categories described before. Given $R^G_i$ such an étale $R^G$-algebra, by base change we get that $R \to R \otimes_{R^G} R^G_i$ is a $G$-equivariant étale $R$-algebra. Conversely, given $R_i$ such a $G$-equivariant étale $R$-algebra, by universal property of the rings of invariants, we get an induced étale morphism $R^G \to R_i^G$.

□

**Proposition 3.36.** *Let $S$ be a locally noetherian scheme and $V$ a nodal curve over $S$ with an action of the group of roots of unity $\mu_r$ compatible with the local picture. Then $[V/\mu_r]$ is a twisted curve.*

*Proof.* Let $\overline{v} \to [V/\mu_r]$ be a geometric point. Without loss of generality, suppose that $V = Spec(B)$ is affine; then the coarse moduli space of $[V/\mu_r]$ is $Spec(B^{\mu_r})$. Since $V$ is a nodal curve, its local picture at $\overline{v}$ is given by $Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)$, for $\alpha \in A$, at a suitable geometric point. We need to prove that:

$$[V/\mu_r]^{sh} \cong \left[Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)/\mu_r\right]^{sh}$$

which is equivalent to:

$$\left[V \times_{Spec(B^{\mu_r})} Spec(B^{\mu_r})^{sh}/\mu_r\right] \cong \left[Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right) \times_{Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)^{\mu_r}} \left(Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)^{\mu_r}\right)^{sh}/\mu_r\right]$$

But $V^{sh} \cong Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)^{sh}$ and by Lemma 3.35:

$$V \times_{Spec(B^{\mu_r})} Spec(B^{\mu_r})^{sh} \cong Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right) \times_{Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)^{\mu_r}} \left(Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)^{\mu_r}\right)^{sh}$$

since they have the same description in terms of direct limit of $\mu_r$-equivariant étale algebras. To conclude, the action on the nodal curve is compatible with its local picture, so we get the required isomorphism. □

By the propositions, we see that a twisted curve $\mathcal{C} \to S$ has a coarse moduli space $C \to S$ which is a nodal curve and is locally étale the quotient of a nodal curve $V \to S$ by the action of a group of roots of unity $\mu_r$ given in (3.1). Moreover, we can assume that $V = Spec(B)$ is affine. Let's see the relation between these two nodal curves, $V$ and $C$. Suppose that the local picture of $V \to S$ at a geometric point is given by:

$$Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right) \to Spec(A)$$

Since étale locally $\mathcal{C}$ is the quotient $[V/\mu_r]$, the coarse moduli space $C$ is étale locally $Spec\left(B^{\mu_r}\right)$. But as already described:

$$B^{\mu_r} = \frac{A[v,w]}{(vw-\alpha^r)} \to \frac{A[x,y]}{(xy-\alpha)}$$

is the map presenting the affine quotient on the rings, sending $v \to x^r$ and $w \to y^r$. Let's conclude with a description of the groups of automorphisms of the geometric points of a twisted curve.

**Proposition 3.37.** *Let $S$ be a locally noetherian scheme and $\mathcal{C}$ a twisted curve over $S$. Then all smooth geometric points of $\mathcal{C}$ have trivial automorphisms group.*

*Proof.* Let the local picture of $\mathcal{C}$ be given by:

$$\left[Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)/\mu_r\right] \to Spec(A)$$

at a certain point. Then if $\alpha \notin m_A$, by the jacobian criterion all the points are smooth, and none of them correspond to an ideal containing $x, y$, hence as described before the group of automorphisms is trivial. If $\alpha \in m_A$, it can have a non-trivial automorphism group if it contains $x, y$, but then it should contain also $m_A$, contrary to the hypothesis of being smooth.

□

## 3.4 Characterization of Twisted Curves

In this section, we want to generalize the characterization we proved for nodal curves in Corollary 3.26 in terms of their geometric fibers to twisted curves. In the way, we recall and prove some facts on the behaviour of strict henselization.

**Lemma 3.38.** *Let $R$ be a local ring with strict henselization $R^{sh}$. Let $I \subseteq m_R$ be an ideal. Then:*

$$(R/I)^{sh} \cong R^{sh}/IR^{sh}$$

*Proof.* See [15] Stacks Project, Tag 05WS. □

We start by showing that the geometric fibers of a twisted curve are twisted curves.

**Proposition 3.39.** *Let $S$ be a locally noetherian scheme and $\mathcal{C} \to S$ a twisted curve over $S$. Then every geometric fiber of $\mathcal{C} \to S$ is a twisted curve.*

*Proof.* By hypotheses, $\mathcal{C} \to S$ is a twisted curve, hence, at every geometric point of $\mathcal{C}$, its local picture is given by:

$$\left[Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)/\mu_r\right] \to Spec(A)$$

for $\alpha \in A$. Let $Spec(k) \to S$ be a geometric point of the base scheme $S$, and $\mathcal{C}_k$ the geometric fiber. Denote by $U = \frac{A[x,y]}{(xy-\alpha)}$. By hypotheses:

$$\mathcal{C} \times_C C^{sh} \cong [Spec(U)/\mu_r] \times_{Spec(U^{\mu_r})} Spec(U^{\mu_r})^{sh}$$

and taking the fiber:

$$\mathcal{C}_k \times_{C_k} C^{sh} \times Spec(k) \cong [Spec(U \otimes k)/\mu_r] \times_{Spec((U\otimes k)^{\mu_r})} Spec(U^{\mu_r})^{sh} \times Spec(k)$$

But by Lemma 3.38, $C^{sh} \times Spec(k) \cong (C \times Spec(k))^{sh}$, since tensoring with $k$ is an operation involving the quotient. We conclude:

$$\mathcal{C}_k \times_{C_k} (C_k)^{sh} \cong [Spec(U \otimes k)/\mu_r] \times_{Spec((U\otimes k)^{\mu_r})} Spec(U^{\mu_r} \otimes k)^{sh}$$

that makes the geometric fiber $\mathcal{C}_k$ into a twisted curve. □

As for the case of nodal curves, the hardest part is characterizing twisted curve over an excellent scheme $S$ knowing that the geometric fibers are twisted curves.

**Lemma 3.40.** *Let $A$ be a local noetherian henselian ring and $R$ be the henselization of $\frac{A[x,y]}{(xy-\alpha)}$ at the ideal $(x, y, m_A)$, for $\alpha \in m_A$. Let $\overline{x}, \overline{y}$ be the images of $x, y$ in $R$. Suppose that we are given $x', y' \in R$, $\alpha' \in A$ such that $x'y' = \alpha'$ and $(\overline{x}, \overline{y}, m_A) = (x', y', m_A)$. Then there exist units $u, v \in R^\times$ such that $uv \in A^\times$ such that either $u\overline{x} = x'$ and $v\overline{y} = y'$ or $u\overline{x} = y'$ and $v\overline{y} = x'$.*

*Proof.* See Kato [9] Lemma 2.1. □

**Lemma 3.41.** *Let $A$ be a ring, $m$ a prime ideal of $A$ and $\alpha \in m$. Denote by $R$ the strict henselization of $\frac{A[x,y]}{(xy-\alpha)}$ at the ideal $(x, y, m)$. Then $R$ is isomorphic to the strict henselization of $\frac{A^{sh}_m[x,y]}{(xy-\alpha)}$ at the ideal $(x, y, m)$.*

*Proof.* By Lemma 3.38, taking the strict henselization commutes with taking the quotient, so we just have to prove that:

$$(A[x,y])^{sh} \cong \left(A^{sh}[x,y]\right)^{sh}$$

where by $A^{sh}$ we denote the strict henselization of $A_m$ and by $(A[x,y])^{sh}$ the strict henselization at $(x, y, m)$. By universal property of strict henselization:

$$\begin{array}{ccccc}
A^{sh} & \longrightarrow & (A[x,y])^{sh} & \dashrightarrow & \left(A^{sh}[x,y]\right)^{sh} \\
\uparrow & & \uparrow & & \uparrow \\
A & \longrightarrow & A[x,y] & \longrightarrow & A^{sh}[x,y]
\end{array}$$

the dashed arrow exists uniquely. Conversely, by the map $A^{sh} \to (A[x,y])^{sh}$ we get an induced map $\left(A^{sh}[x,y]\right) \to (A[x,y])^{sh}$, sending $x, y$ to the images of $x, y$ of the map $A[x,y] \to (A[x,y])^{sh}$. By universal property:

$$\begin{array}{ccc}
\left(A^{sh}[x,y]\right)^{sh} & \dashrightarrow & (A[x,y])^{sh} \\
\uparrow & & \uparrow \\
A^{sh}[x,y] & \longrightarrow & (A[x,y])^{sh}
\end{array}$$

the dashed arrow exists uniquely. □

We can finally prove the main theorem; the proof will follow the ideas of Olsson in [13] Proposition 2.2.

**Theorem 3.42.** *Let $S$ be an excellent scheme, $\mathcal{C} \to S$ a flat morphism locally of finite presentation with $\mathcal{C}$ a tame Deligne-Mumford stack with finite diagonal, such* *that every geometric fiber of $\mathcal{C} \to S$ is a twisted curve. Then $\mathcal{C} \to S$ is a twisted curve.*

*Proof.* Let $c \to \mathcal{C}$ be a geometric point of $\mathcal{C}$. Since we are interested in the étale local behaviour around $c$, by Theorem 2.47, without loss of generality we may suppose that $\mathcal{C} = [V/G]$, with $V$ a scheme and $G$ a finite group acting on $V$; moreover, $G$ is the group of automorphisms of $c$. By hypothesis, all geometric fibers of $\mathcal{C} \to S$ are twisted curves; in particular, we have that:

$$\mathcal{C}_k = [V_k/G]$$

where $\mathcal{C}_k$ is a geometric fiber of $\mathcal{C} \to S$ and $V_k = V \times Spec(k)$, for a geometric point $Spec(k) \to S$. By Proposition 3.34, we can assume that $V_k$ is a nodal curve. By analogy, all geometric fibers of $V \to S$ are nodal curves and, since $V \to S$ is flat, by Theorem 3.25 $V$ is a nodal curve.
If $c \to \mathcal{C}$ is a smooth point in the fiber, by Proposition 3.37 the group $G$ is trivial, hence:

$$\mathcal{C} = [V/G] \cong V$$

So if the local picture of $V$ was given by $Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)$ with $\alpha \in A$, for a ring $A$, then it gives also the local picture of $\mathcal{C}$.
If $c \to \mathcal{C}$ is not a smooth point in the fiber, then the local picture of $V$ is given by $Spec\left(\frac{A[x,y]}{(xy-\alpha)}\right)$ at a geometric point lying over $(x, y, m_A)$ with $\alpha \in m_A$, where $m_A$ is the ideal corresponding to the geometric point $Spec(k) \to S$. It is not restrictive to suppose $V = Spec(B)$ is affine, hence there is an isomorphism:

$$B^{sh} \cong \left(\frac{A[x,y]}{(xy-\alpha)}\right)^{sh}$$

where by $B^{sh}$ we mean the strict henselization of the localization of $B$ at the ideal corresponding to the geometric point $c$, and by $\left(\frac{A[x,y]}{(xy-\alpha)}\right)^{sh}$ the strict henselization of the localization of $\frac{A[x,y]}{(xy-\alpha)}$ at $(x, y, m_A)$. The fiber $\mathcal{C}_k$ is a twisted curve, hence the local picture of $V_k$ is given by $Spec\left(\frac{k[x,y]}{(xy)}\right)$ and, if we denote by $\overline{B} = B \otimes_{\mathcal{O}_S} k$ the reduction to the geometric fiber, there is an action of $G$ on $\overline{B}^{sh}$ such that the isomorphism:

$$\overline{B}^{sh} \cong \left(\frac{k[x,y]}{(xy)}\right)^{sh}$$

is $G$-equivariant. Finally, since $\mathcal{C}_k$ is a twisted curve, we know that $G$ acts as a group of $r$-roots of unity sending $x \to \xi x$ and $y \to \xi^{-1} y$, where $\xi$ is a primitive $r$-root of unity. Fix a generator $\gamma \in G$. Denote by $z, w, \beta$ the images of $x, y, \alpha$ in $B^{sh}$ and by $\overline{z}, \overline{w}$ their reduction in $\overline{B^{sh}}$ (recall that, by Lemma 3.38, $\overline{B}^{sh} \cong \overline{B^{sh}}$). Since $\beta$ is $G$-invariant, we have that $\gamma(z)\gamma(w) = \beta$ and, in particular, $(z, w, m_A) = (\gamma(z), \gamma(w), m_A)$ (equality of ideals in $B^{sh}$; by a little abuse, we use $m_A$ instead of its image in $B^{sh}$). On the fiber, the action of $G$ preserves the two components of $Spec\left(\frac{k[x,y]}{(xy)}\right)$. By Lemma 3.41 we can replace $A$ with its strict henselization $A^{sh}$.

Moreover, since the residue field of (the localization of) $\frac{A^{sh}[x,y]}{(xy-\alpha)}$ is separably closed, we have that its henselization is equal to its strict henselization. Now by Lemma 3.40 we get that $\gamma(z) = uz$ and $\gamma(w) = vw$ for $u, v \in \left(B^{sh}\right)^{\times}$. $G$ is a finite group of order $r$, hence $u, v$ are $r$-roots of unity; in particular, since on the fiber the action sends $x \to \xi x$ and $y \to \xi^{-1}y$, we should have also $u = \xi$ and $v = \xi^{-1}$. To conclude, we have endowed the nodal curve $V$ with an action of a group of roots of unity compatible with the local picture, so by Proposition 3.36 $\mathcal{C}$ is a twisted curve. □

**Corollary 3.43.** *Let $S$ be an excellent scheme and $\mathcal{C} \to S$ be a flat morphism locally of finite presentation with $\mathcal{C}$ a tame Deligne-Mumford stack with finite diagonal. Then the following are equivalent:*

*(a) $\mathcal{C} \to S$ is a twisted curve,*

*(b) all the geometric fibers of $\mathcal{C} \to S$ are twisted curves.*

*Proof.* Combine Proposition 3.39 and Theorem 3.42. □

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